Start Project Y 192
hp - Finite Elements: Fast Solvers and Adaptivity
Annual Report 2002

Abstract

The start project “hpFEM” was started in August 2002 by Dr. Joachim Schöberl. Dipl.-Ing. Almedin Bećirović and Dipl.-Ing. Robert Gaisbauer joined in September, Dipl.-Ing. Sabine Zaglmayr came to the team in November.

In this period, we extended the existing finite element package NGSolve to high order finite elements. Now, we can solve two and three dimensional heat flow and elasticity problems, and two dimensional magnetic field problems with arbitrary polynomial order. The theoretical focus in this period was on fast solvers for high order methods for scalar problems.

1 Overview

The first step towards an *hp*-finite element method is a high order method. It is called p method, where p indicates the varying polynomial order. The mesh is kept fixed and may consist, as in our case, of triangles for 2D and tetrahedrons for 3D problems. When the solution is smooth, the high order finite elements lead to very fast convergence. If there occur singularities, then the high order method has to be combined with local mesh refinement (called *hp*-method). Our current state is a high order method for 2D and 3D problems. The mesh refinement is left over for the next period.

The following list shows some of the fundamental ingredients of a high order finite element method. These topics were available in the community, and have been implemented into our code. Our new contributions are listed in the following sections.

- **Recursive polynomials**
  The shape functions for the high order finite elements are usually defined by orthogonal polynomials such as Legendre or Gegenbauer polynomials. One advantage is the very efficient computability by a short recursive formula, an other one is the relatively mild growth of the condition number for increasing polynomial order. We have been studying, implementing and testing a variety of recursive polynomials.

- **Mesh topology entities**
  The degrees of freedom, i.e. the unknowns, of high order methods are associated with the mesh entities vertex, edge, face, and interior. The mesh handler has to provide the necessary information, such as element-to-vertex, element-to-edge, face-to-vertex, etc. tables. We have extended the Netgen mesh handler to provide these tables. Also, the orientation of edges and faces is important.

- **High order integration rules**
  The assembling of the high order finite element matrices and vectors require the integration of high order polynomials over the domain of the reference element. We have implemented integration rules of arbitrary order for all types of elements, namely, triangles, quadrilaterals, tetrahedra, pyramids, prisms, and hexahedra by tensor product construction.

- **Visualization**
  The initial code supported visualization of solutions for low order finite elements. We have extended this capability to visualize high order finite element functions. This is done by recursive subdivision of elements.

2 High order curved elements

To obtain high order convergence, also the geometry must be approximated very accurately. We do so by means of curved elements of high polynomial order. R. Gaisbauer has developed and tested methods for the construction of curved elements. Figure 1 shows the mesh of a crank shaft where the element deformation is of order 5.

The mapping of the reference element to the physical one is defined by

$$\Phi(x) = \sum_{Vertex\ v} c_v \Phi_v(x) + \sum_{Edge\ e} \sum_i c_{e,i} \Phi_{e,i}(x) + \sum_{Face\ f} \sum_i c_{f,i} \Phi_{f,i}(x).$$

The scalar functions Φ_v, Φ_e, and Φ_f are finite element basis functions associated to vertices, edges, and faces, respectively. There are several possibilities to define the mapped elements. The first question is the definition of shape functions. In our first attempts, we have chosen bad ones leading to folded elements. Later, we switched to hierarchical shape functions (see Section 3) leading to smoothly mapped elements.

The second question is the choice of the coefficients. The first idea is interpolation in a set of points. It is well known that polynomial interpolation on a uniform grid of points is very unstable, but interpolation in Gauss-Lobatto points is bounded in proper norms. Our experiments confirmed this fact, and we recommend Gauss-Lobatto interpolation on lines and tensor product elements. But, we are interested especially in triangular and tetrahedral elements, where this interpolation does not work anymore, and a new algorithm for the computation of coefficients is needed. In some literature, there is recommended the L_2 best approximation of the true geometry. We came up with another choice, namely the

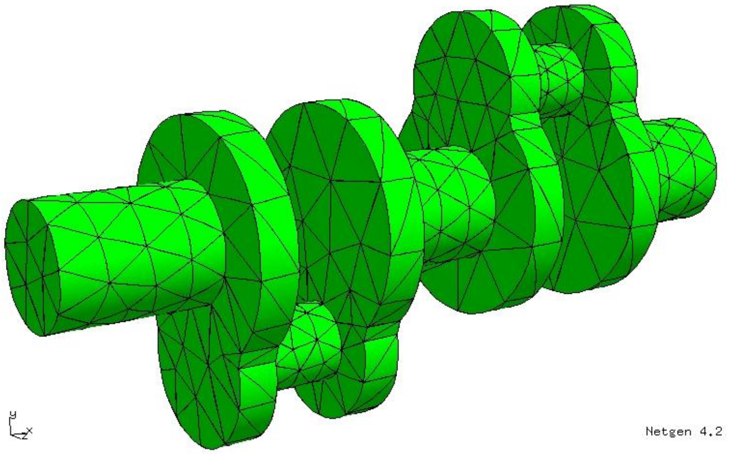

Figure 1: Mesh of crank shaft with curved elements or order 5

	H^1-projection		L_2-projection	
order	err surf	err vol	err surf	err vol
2	3e-2	2e-2	3e-2	1e-2
3	3e-4	3e-4	3e-3	1e-3
4	7e-6	3e-6	2e-4	8e-5
5	7e-8	4e-8	2e-5	8e-6
6	1e-7	7e-8	1e-6	7e-7
7	8e-10	4e-10	2e-7	8e-8
8	6e-10	3e-10	2e-8	8e-9

Table 1: Error in volume and surface of approximated sphere

best approximation in H^1 norm. One reason is the canonical construction of the vertex-edge-face interpolation operator. The other one is that the H^1-norm bounds the maximum norm (in 1D, and nearly 2D), but the L_2-norm does not. We have evaluated the mapped elements by several tests. A simple one is to approximate the surface and the volume of a sphere by integrating over the curved elements. The results in Table 1 show the superior behavior of our new construction of curved elements.

3 Optimal extension shape functions

High order shape functions are associated to vertices, edges, faces, and interior of the elements (in 2D, the faces disappear). This ensures continuity of the global function. E.g., all elements sharing the same edge have the same coefficients for the corresponding edge shape functions. Figure 2 shows some edge and interior shape functions for the triangle.

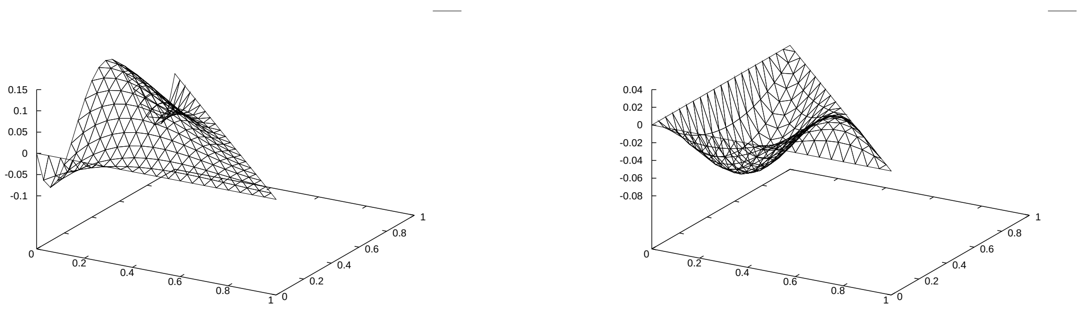

Figure 2: Edge and interior shape functions (p = 4)

The vertex shape functions are just the linear functions. The interior shape functions are all polynomials of prescribed order vanishing at the boundary. The most interesting class is the edge space. At one edge the functions correspond to 1D basis functions, and it has to vanish at all other edges. Their behavior inside the element can be chosen arbitrary - and gives the possibility for tuning.

When assembling the global stiffness matrix, we cluster the unknowns into vertex unknowns, edge unknowns, and interior unknowns. The edge and interior are also clustered edge by edge, and element by element. Then, we apply a block preconditioner with respect to these clusters. If the elements are not too many, we can keep the vertex block as one block, otherwise, one should use one of the h-version preconditioners.

The dependency on the polynomial order p of the preconditioner can be studied on a mesh consisting of one element. The matrix clustered into the blocks is

$$A_T = \begin{pmatrix} A_{VV} & A_{VE} & A_{VI} \\ A_{EV} & A_{EE} & A_{EI} \\ A_{IV} & A_{IE} & A_{II} \end{pmatrix},$$

the block Jacobi preconditioner is

$$C_T = \begin{pmatrix} A_{VV} & 0 & 0 \\ 0 & \widetilde{A}_{EE} & 0 \\ 0 & 0 & A_{II} \end{pmatrix}.$$

The interactions of the individual blocks are neglected. We measure the relative condition number $\kappa\{C_T^{-1} A_T\}$. The optimal value would be 1. The goal is simple to explain: Choose the basis such that the off diagonal entries are small. The focus of the work of A. Bećirović is to define the edge shape functions such that the coupling term A_{EI} is as small as possible. We have made many tests for quadrilaterals, and have obtained a considerable improvement to the standard approach. For triangles, we have found shape functions leading to condition

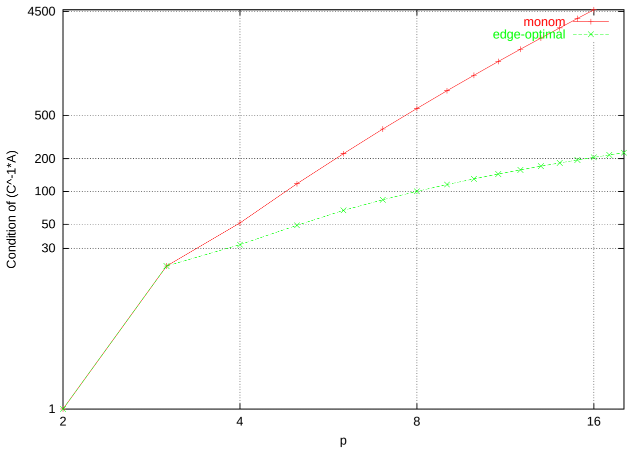

Figure 3: Condition numbers with monomial shape functions and optimal edge shape functions for tetrahedra

numbers of optimal order. The same edge shape functions are nearly optimal when used as tetrahedra edge shape functions. Figure 3 shows the comparison of condition numbers for the standard monomial shape functions and the new ones for tetrahedra. The construction of optimal face shape functions for tetrahedra is now clear, but not yet implemented. By now, we use the standard extension. Figure 4 below shows the mechanical stresses in a screw-wrench. The simulation used 2147 elements of 5^{th} order.

4 Commuting shape functions for $H(\mathrm{div})$ and $H(\mathrm{curl})$

Electromagnetic field simulations are based on the vector valued function spaces $H(\mathrm{div})$ and $H(\mathrm{curl})$. Many properties of the finite element discretization (such as interpolation operators, preconditioners, a posteriori estimates etc) are tightly connected to the commuting diagram:

$$\begin{tikzcd}
H^1 \arrow[r, "\nabla"] \arrow[d, "\Pi^W"] & H(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\Pi^Q"] & H(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi^V"] & L^2 \arrow[d, "\Pi^W"] \\
W_h \arrow[r, "\nabla"] & Q_h \arrow[r, "\mathrm{curl}"] & V_h \arrow[r, "\mathrm{div}"] & S_h\,.
\end{tikzcd} \tag{1}$$

In short, the diagram shows the relation of the function spaces in the first line, and corresponding relations of the finite element spaces in the second line. S. Zaglmayr developed high order finite elements for $H(\mathrm{div})$ and $H(\mathrm{curl})$, and implemented the 2D case. These new shape functions are commuting in the following sense: The gradients of H^1 edge shape functions are $H(\mathrm{curl})$-edge-shape functions. The same hold for face shape functions, and

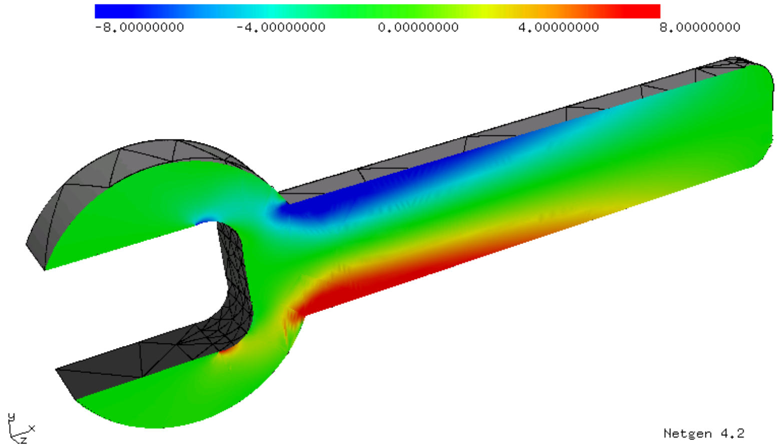

Figure 4: Stresses in a screw-wrench

also, the curl of $H(\text{curl})$ face shape functions are $H(\text{div})$ face shape functions. The advantage of the new construction is that simple preconditioners work well. Some of these shape functions are drawn in Figure 5.

The $H(\text{curl})$ finite elements have been used for a 2D computation of a C - magnet, see Figure 6. A coil around the limb (on the left hand side) drives the magnetic flux in the magnet. The flux spreads out in the air gap on the right hand side. The simulation used elements of order 8.

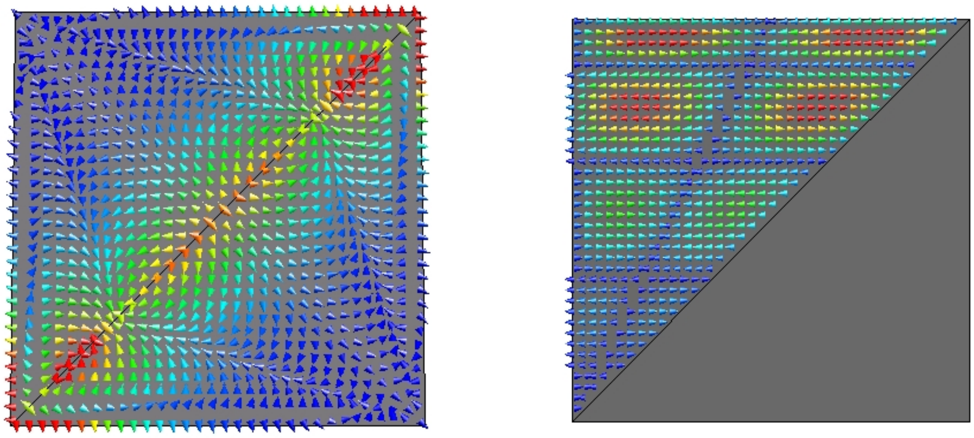

Figure 5: Edge shape function ($p = 5$) and inner shape function ($p = 8$)

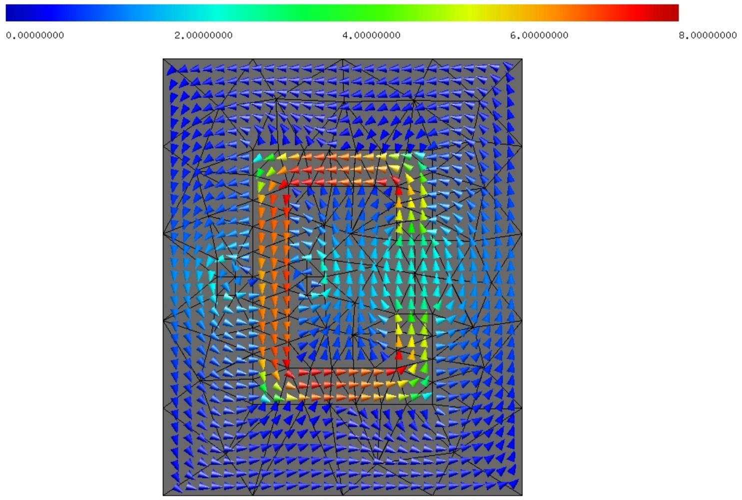

Figure 6: Magnetic flux in the C-magnet