Finite Element Methods

The finite element method approximates solutions of partial differential equations by piecewise polynomial functions on simple geometric elements. In the h version of the FEM, convergence is obtained by decreasing the element size h. The p version keeps the mesh fixed, but increases the polynomial order p. The combination of both, called hp-FEM, combines high order in smooth regions with local mesh refinement at singularities, it leads to exponentially fast convergence.

High Order Shape functions

High order shape functions are associated with the logical nodes vertex, edge, face (3D only), and interior. One starts with a polynomial basis for the interval. Hierarchical orthogonal polynomials are cheap to evaluate, and the resulting condition numbers grow moderately with p.

The construction for 2D and 3D elements is based on tensor products. For example, inner shape functions for triangles are defined as

\[ \phi_{ij} = (1 - x)P_1(\frac{y}{1-x}) \]

with \(0 \leq i, j \leq 3, p = 3\).

The VEFI - construction ensures the proper continuity requirements.

Curved Elements

To obtain high order convergence, also the geometry must be approximated very accurately. We do so by means of curved elements of high order.

The mapping of the reference element to the physical one is defined by

\[ z = \sum_{\alpha=1}^{\text{np}} \phi_{\alpha}^b(\xi) + \sum_{\alpha=1}^{\text{np}} \sum_{\beta=1}^{\text{ep}} \sum_{\gamma=1}^{\text{sp}} \sum_{\delta=1}^{\text{dp}} W_{\alpha\beta\gamma\delta}(\xi) \Phi_{\alpha\beta\gamma\delta}(\eta) \]

The scalar functions \( \phi_{\alpha}, \Phi_{\alpha\beta\gamma\delta} \) are finite element basis functions associated to vertices, edges, and faces, respectively. The coefficient vectors \( V_{\alpha}, W_{\beta\gamma}, \) and \( W_{\alpha\beta\gamma\delta} \) need to be computed for each global element separately.

Besides choosing suitable basis functions, another problem is how to obtain the coefficient vectors \( W_{\alpha\beta\gamma\delta} \) and \( W_{\beta\gamma} \). We do so by projecting the exact geometry edges and faces onto the space of polynomials in the \( H^1 \) semi-norm. This leads to superior results that projecting in the \( L^2 \) norm.

Domain Decomposition Preconditioning

We want to apply iterative solvers such as the preconditioned Richardson iteration

\[ u = u + \alpha (f - Au) \]

or the conjugate gradient iteration. To obtain an efficient method, the preconditioner \( C \) must be a good approximation to \( A \), and the operation \( C^{-1} \) must be cheap. A simple preconditioner is the Block Jacobi one:

\[ A = \begin{pmatrix} A_{vv} & A_{vx} & A_{vx} \\ A_{vx} & A_{xx} & A_{vx} \\ A_{vx} & A_{vx} & A_{xx} \end{pmatrix}, \quad C = \begin{pmatrix} A_{vv} & 0 & 0 \\ 0 & A_{xx} & 0 \\ 0 & 0 & A_{xx} \end{pmatrix} \]

The condition number depends on the chosen basis functions.

Explicit Extension Operators

We define the edge shape functions by optimal extension operators from the boundary into the triangle. It suffices to consider one edge.

Step 1: Perform local averaging.

[Babuška + Ciaug + Mandel + Prikára], [91]

\[ u(x, y) = \frac{1}{2} \int_{y=x}^{y=x+1} u(x, s) \, ds \]

This extension preserves the polynomial order, and is a bounded operator in \( H^{1/2}(E) \to H^1(T) \).

Step 2: Maintain zero boundary values at upper 2 edges by linear interpolation [Musé & Sola, 97]. The upper right edge is eliminated by

\[ u(x, y) = u(x, y) - \frac{1}{2} \int_{x+y=1} u(x, y) \, ds \]

The operator for the upper left edge is similar.

This operator is bounded in \( H^{1/2}(E) \to H^1(T) \).

The extension operator must be applied to polynomial basis functions on the edge. P. Paul et al. could apply their hypergeometric summation algorithms to derive a convenient recurrence relation for \( u(x, y) \):

\[ u_{ij} = a_{ij} u_{i-1,j} + b_{ij} u_{i,j-1} - (k^2 - y^2) u_{i,j} + e_{i,j} u_{i-1,j-1} \]

The coefficients \( a, b, c, d, e \) are rational in \( l \) and are computed once and for all.

Numerical Results

Applying the DD preconditioner to one element matrix suffices to investigate the asymptotic behavior in the polynomial order. We have compared the condition numbers when using standard shape functions against the optimal extension shape functions:

Maxwell Equations

Let \( \mathbb{O} \subset \mathbb{R}^3 \). The time-harmonic Maxwell equations are stated by

\[ \nabla \times E = j / \omega \mu \quad \nabla \times H = i / \omega \varepsilon \varepsilon \]

with \( E, \) electric field intensity

\( H, \) magnetic field intensity

\( j, \) impressed current density

Magntoebiste BVP for vec. pot. \( A, H = \mu^{-1} \nabla \times A \)

Find \( A \in H(\text{curl}, \Omega) \). \( s, t, \forall v \in H(\text{curl}, \Omega) : \)

\[ \int \nabla \times v \times \nabla \times A \, ds = \int \nabla \times v \times j \, ds \]

High Order H(\text{Curl}) Finite Elements

De Rham Complex for continuous and discrete spaces

\[ H^1 \to H(\text{curl}) \to H(\text{div}) \to H^2 \]

\[ U \cup U \cup U \]

\( W_{h+1} \to \sum \quad V_{h} \cup V_{h} \to \sum \quad Q_{h,0} \cup Q_{h,0} \]

\( W_{h+1} \to \sum \quad V_{h} \cup V_{h} \to \sum \quad Q_{h,0} \cup Q_{h,0} \)

with complete sequence property

\[ \text{sequence}(\nabla) \to \text{sequence}(\text{curl}) \]

for continuous and discrete cases.

Construction of hierarchical Vertex-Edge-Face-Interior bases satisfying

\[ W_{h+1} \to \sum \quad V_{h} \cup V_{h} \to \sum \quad Q_{h,0} \cup Q_{h,0} \]

Advantages of using hierarchical basis satisfying complete sequence property

- allows variable independent order of \( p_x, p_y, p_z \)
- E-F-I Block-Jacobi is efficient
- saving dofs for magnetostatic problems via 'Gauging'
- efficient calculation of gradient matrix \( B \)

Some \( H_{\text{curl}}(\text{curl}) \) shape functions on triangles

Numerical results for magnetostatic BVP

Maxwell Eigenvalue Problem

We want to solve for resonance frequencies of the time-harmonic Maxwell equations. The EVP in weak form is

\[ \int \nabla \times v \times \nabla \times A \, ds = \int \nabla \times v \times j \, ds \quad \forall v \in (H(\text{curl}, \Omega)) \]

In view of the discretization we have

\[ A_{\text{eigenvalue}} = \lambda \mu \mathbf{A} \quad \text{with} \quad \text{det}(\mathbf{A}) = -\nabla \times \mathbf{A} \]

Inexact inverse iteration with inexact projection

- Inexact inverse iteration:
  \[ u_{n+1} = u_n - C^{-1}\nabla^2 \nabla^2 (A - \lambda_{n} \mathbf{M}_{\text{eigenvalue}}) \]
  \[ \text{Exact L}^2 \text{-projection of complement of } \mathbf{A}_{\text{eigenvalue}} = \mathbf{B}_{\text{proj}} \text{ with } \mathbf{B}_{\text{proj}} \to \mathbf{W}_{\text{proj}} \to \mathbf{V}_{\text{proj}} \text{ and solving Poisson-Problem} \]
  \[ \nabla \nabla \mathbf{V}_{\text{proj}} = (\mathbf{B}_{\text{proj}} + \mathbf{B}_{\text{proj}}^T) \mathbf{V}_{\text{proj}} \quad \forall \mathbf{V}_{\text{proj}} \to \mathbf{W}_{\text{proj}} \]
  \[ \text{is approximated by } k \text{ inexact projection steps} \]
  \[ u_{n+1} = (I - \mathbf{P}_{\text{proj}} C_{\text{proj}}) \mathbf{M}_{\text{eigenvalue}} \]
  \[ \text{with } H(\text{curl})\text{-preconditioner } C_{\text{proj}} = K_{\text{proj}} \]

\[ K_{\text{proj}} \]

Attractive \( \text{poly} \)