Approximation of Curvilinear Bounded Domains by Polynomial Curved Finite Elements in 2 and 3 Dimensions

Robert Gaisbauer
FWF Start - Project "hp-FEM"
Department of Computational Mathematics
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Contents:

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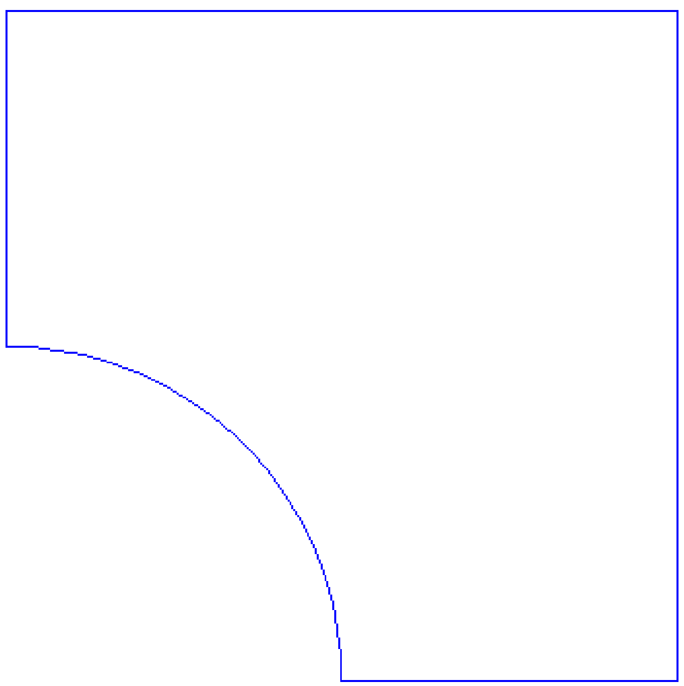

curved domain

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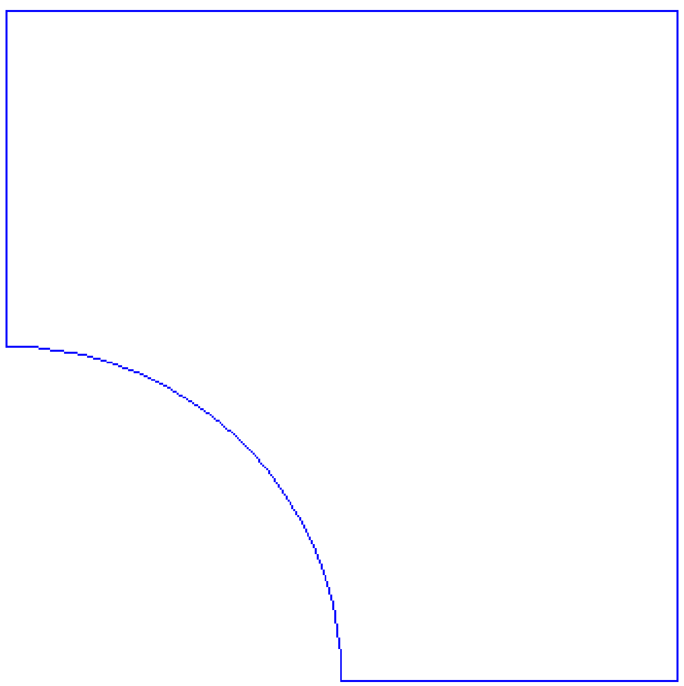

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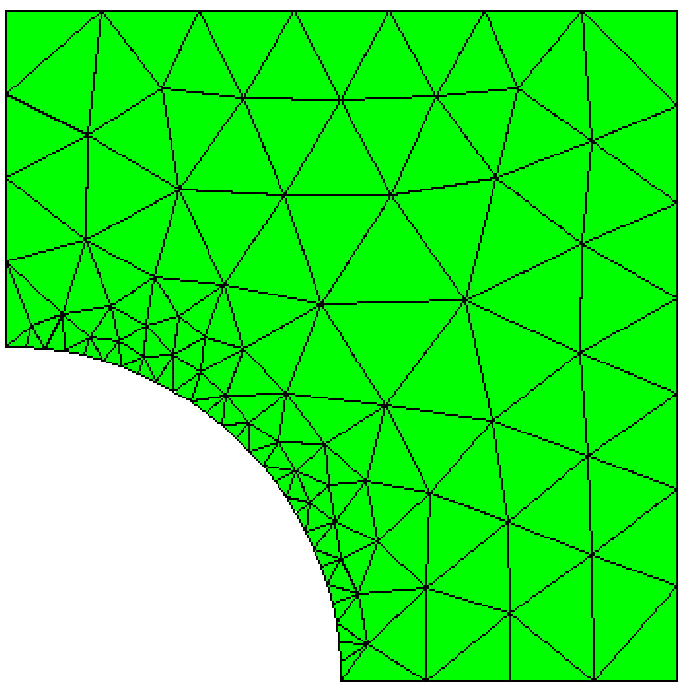

linear elements $(p = 1)$
$N_E = 151$

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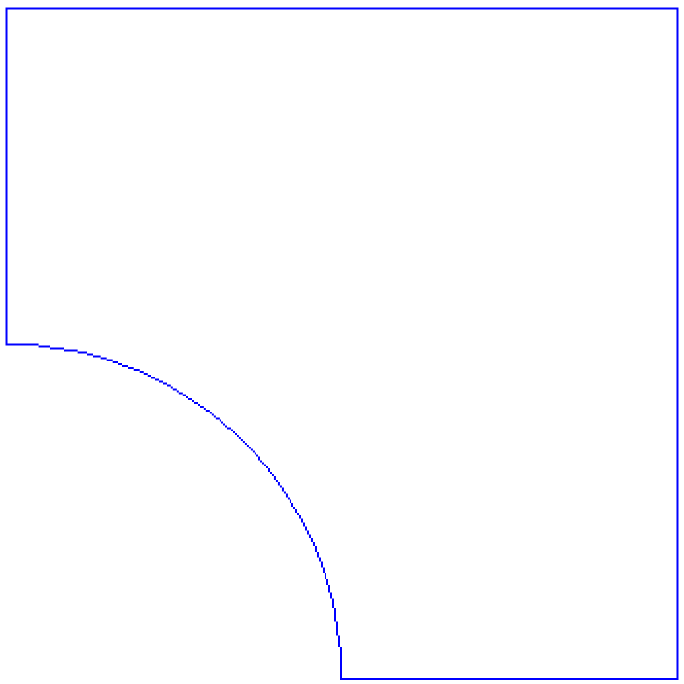

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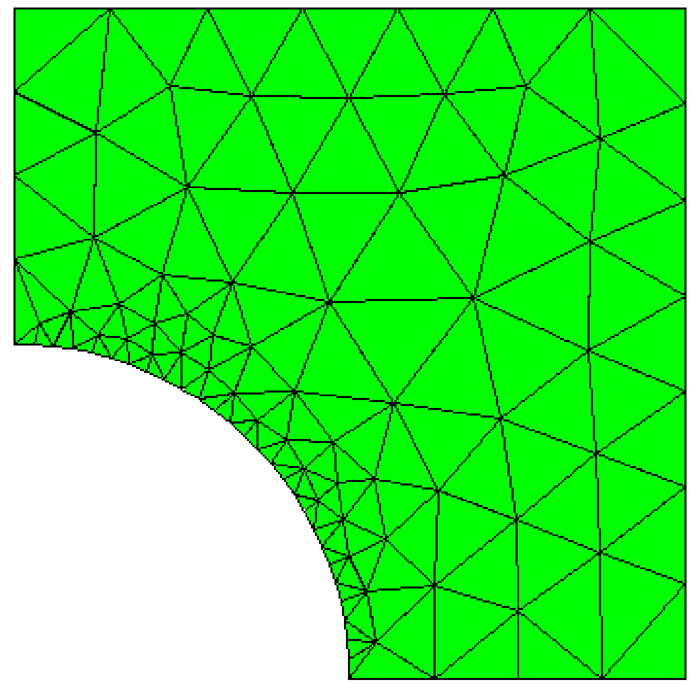

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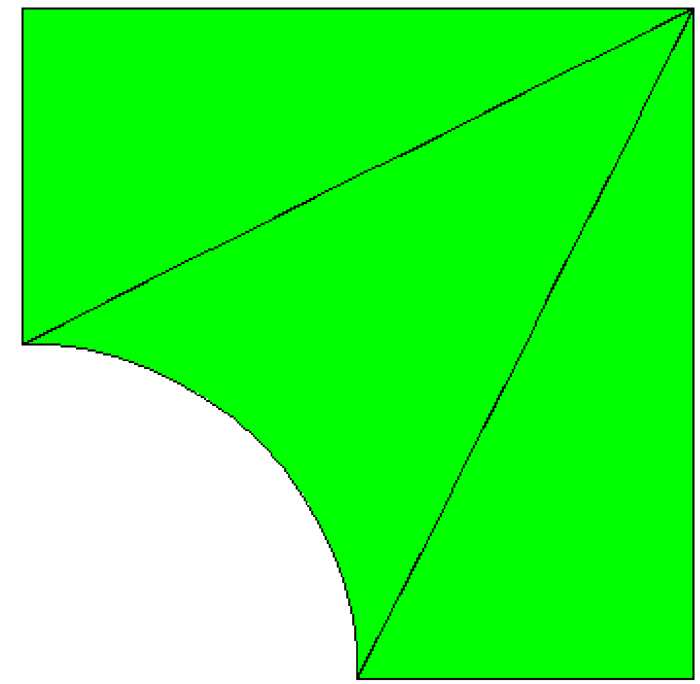
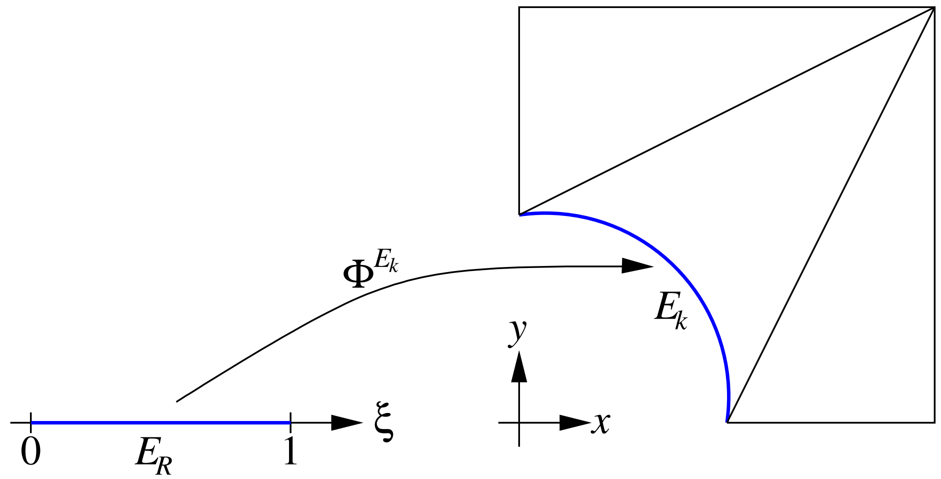


curved elements ($p > 1$)
$N_E = 3$

2D: What do we need?

Boundary integrals (edge mapping)



$$\int_{E_k} f(x) \cdot dx = \int_{E_R} f(\Phi^{E_k}(\xi)) \left| \frac{\partial \Phi^{E_k}}{\partial \xi} \right| d\xi$$

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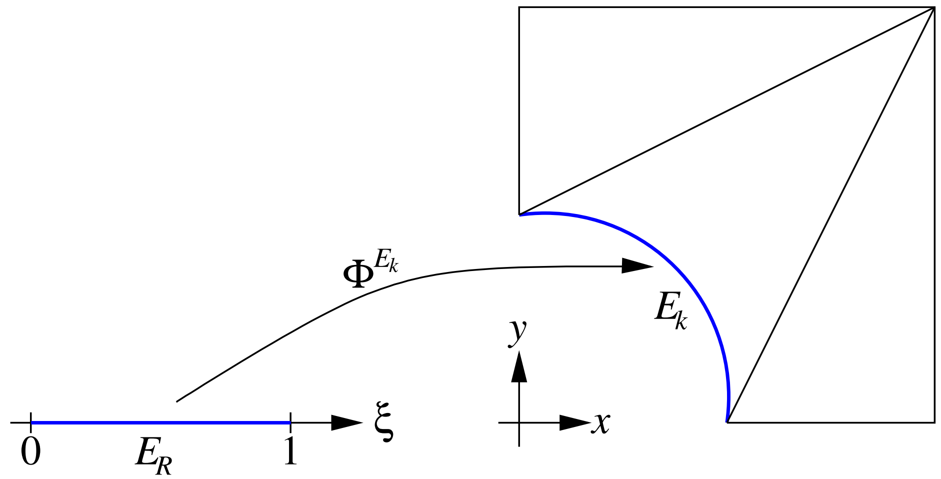

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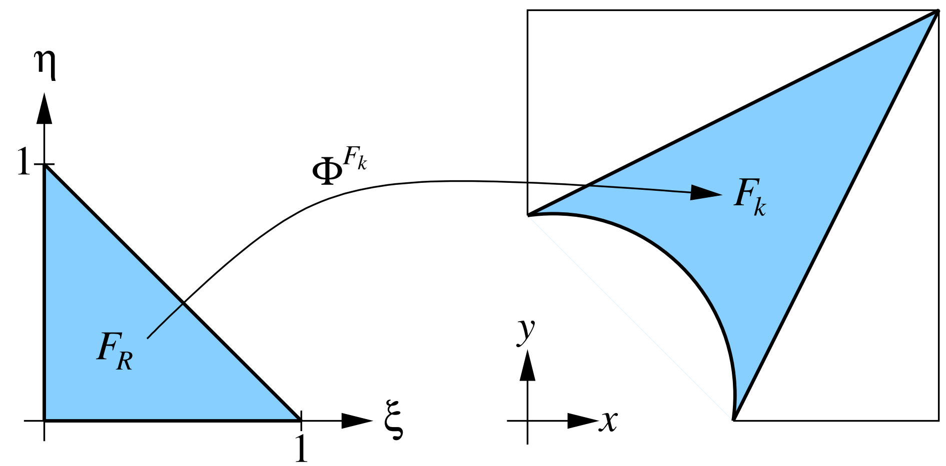
Volume integrals (face mapping)



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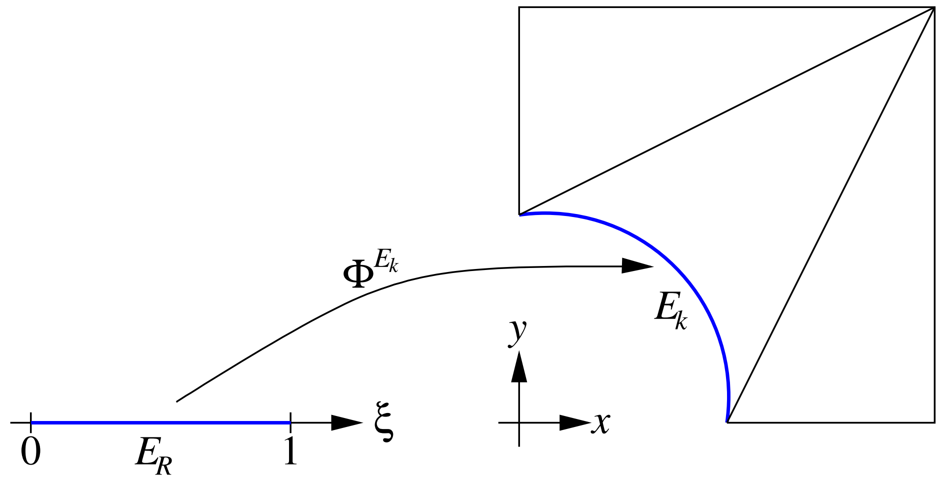

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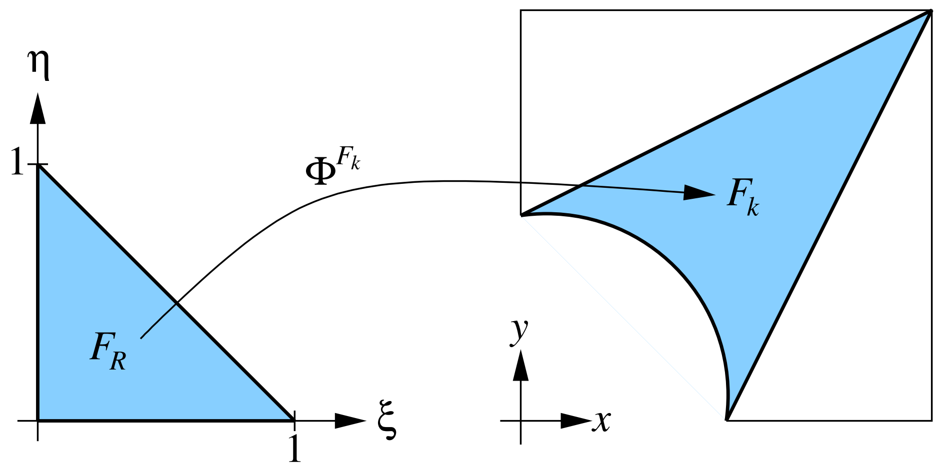

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Construct:

$$\Phi^{E_k}, \frac{\partial \Phi^{E_k}}{\partial \xi}$$

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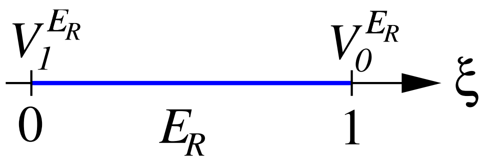
A usefull tool: barycentric coordinates



$\xi = \sum_{i=0}^{1} \lambda_i V_i^{E_R}$

$\sum_{i=0}^{1} \lambda_i = 1$

$\lambda_0 = \xi$
$\lambda_1 = 1 - \xi$

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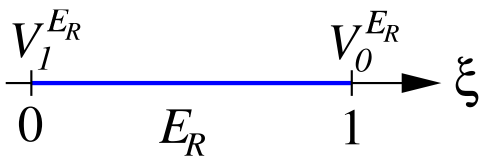

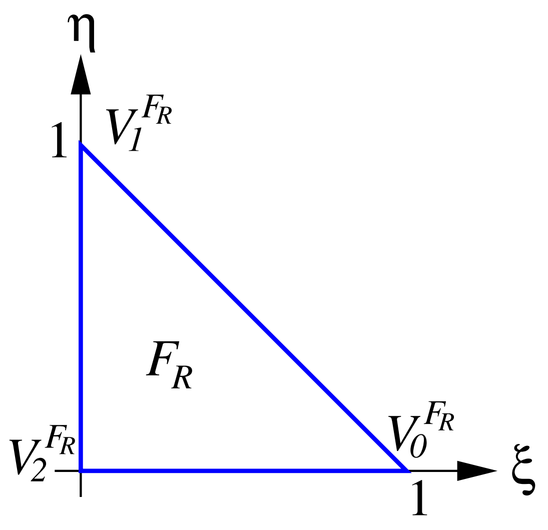

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$\lambda_0 = \xi$
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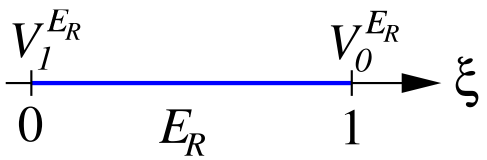
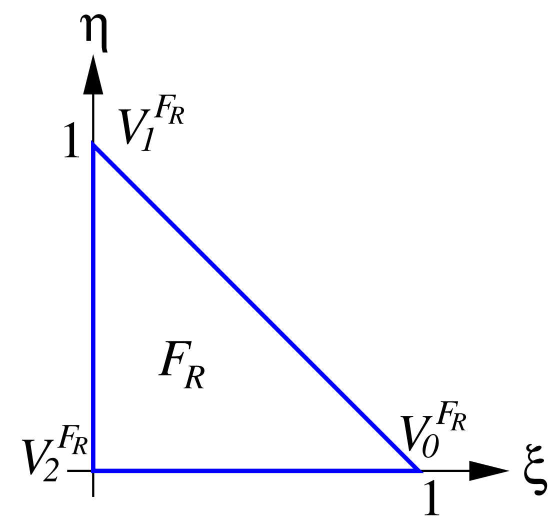
$(\xi, \eta) = \sum_{i=0}^{2} \lambda_i V_i^{F_R}$

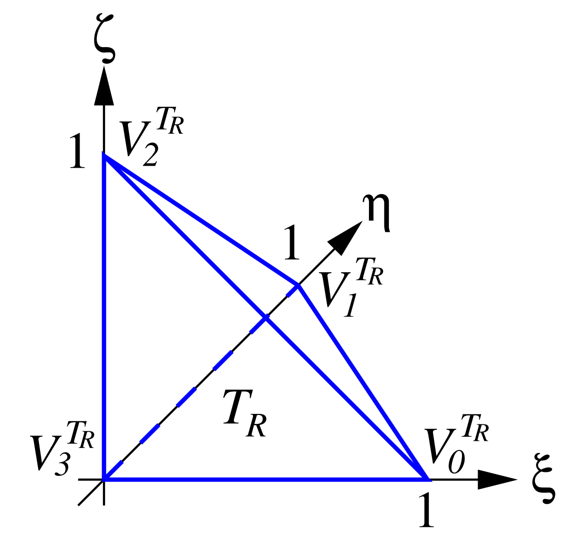
$\sum_{i=0}^{2} \lambda_i = 1$

$\lambda_0 = \xi$
$\lambda_1 = \eta$
$\lambda_2 = 1 - \xi - \eta$

A usefull tool: barycentric coordinates







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$\lambda_2 = 1 - \xi - \eta$

$(\xi, \eta, \zeta) = \sum_{i=0}^{3} \lambda_i V_i^{T_R}$

$\sum_{i=0}^{3} \lambda_i = 1$

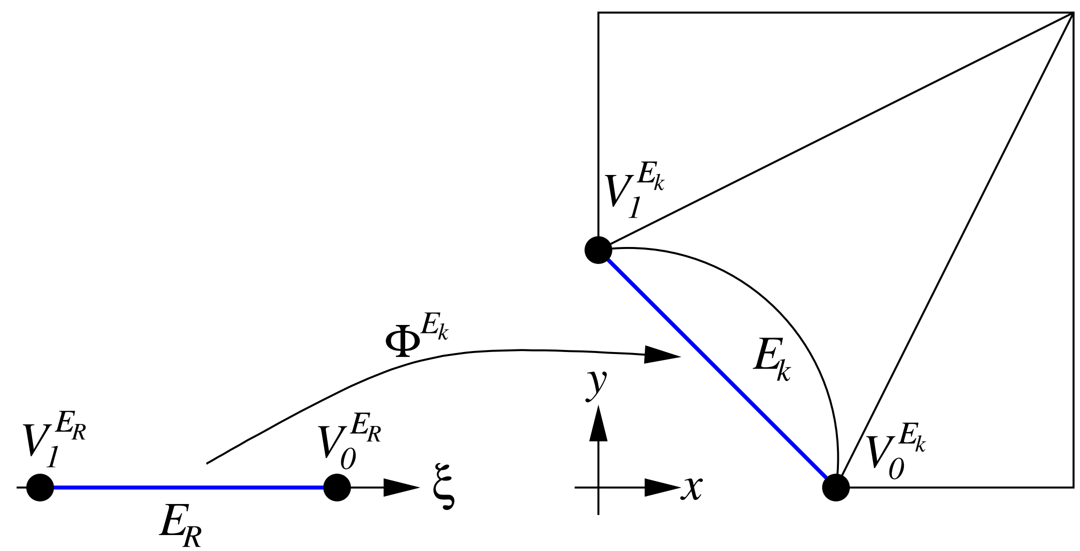
$\lambda_0 = \xi$
$\lambda_1 = \eta$
$\lambda_2 = \zeta$
$\lambda_3 = 1 - \xi - \eta - \zeta$

Structure of the mappings (2D, edge mapping)

$$\Phi^{E_k}(\xi) = \sum_{l=0}^{1} \Phi^{E_k,l}_{vertex}(\xi)$$

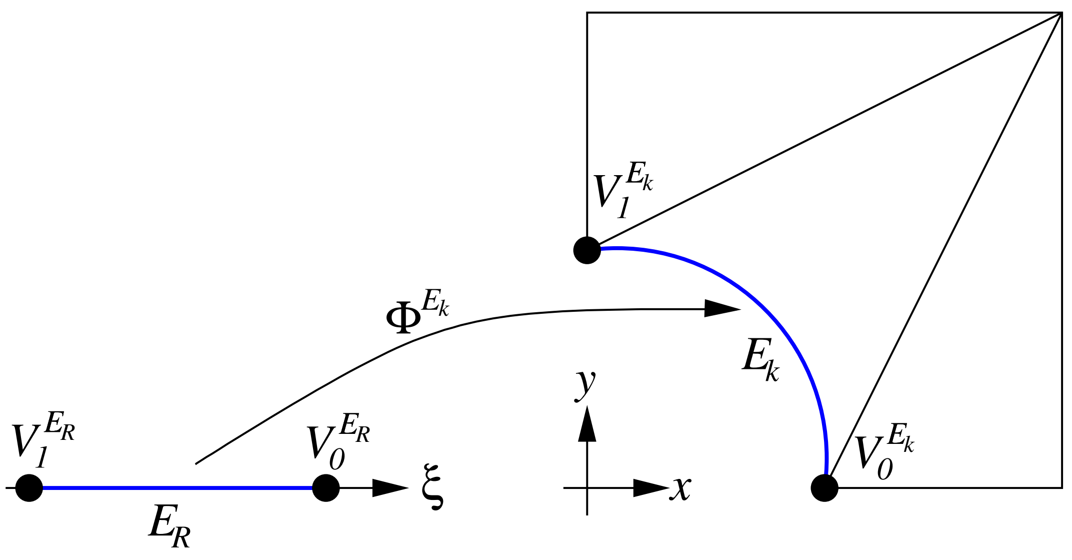
with

$$\Phi^{E_k,l}_{vertex}(\xi) \quad = \quad \lambda_l V_l^{E_k}$$



Structure of the mappings (2D, edge mapping)

$$\Phi^{E_k}(\xi) = \sum_{l=0}^{1} \Phi^{E_k,l}_{vertex}(\xi) + \Phi^{E_k}_{edge}(\xi)$$



with

$$\begin{aligned} \Phi^{E_k,l}_{vertex}(\xi) &= \lambda_l V_l^{E_k} \\ \Phi^{E_k}_{edge}(\xi) &= \sum_{i=2}^{p} W_i^{E_k} b_i(\lambda_0) \end{aligned}$$

and

$W_i^{E_k} \in \mathbf{R}^2$
$\{b_i\}_{i=2,\ldots p}$ basis of $\mathbf{P}_p^0(E_R)$

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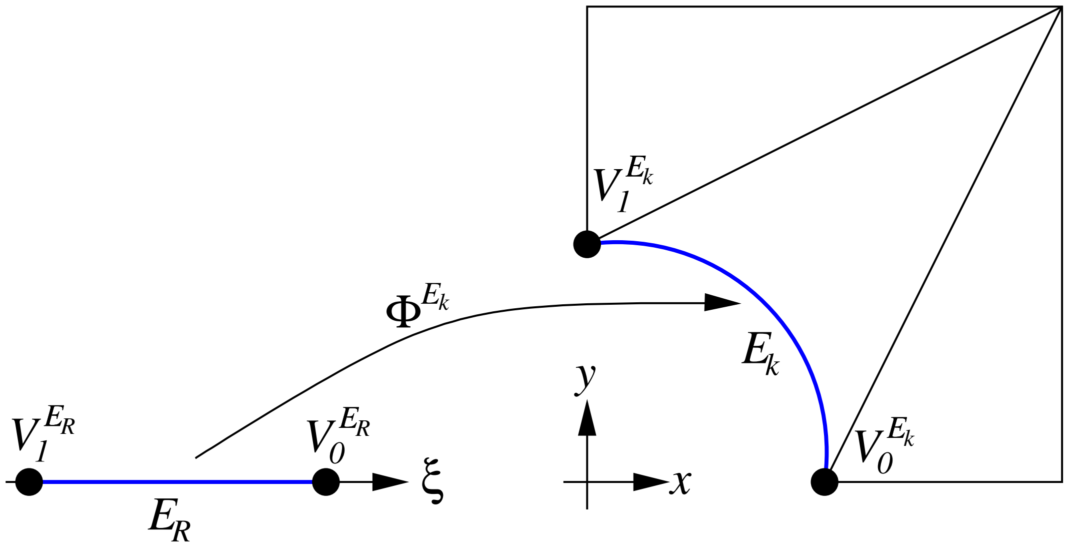

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$\{b_i\}_{i=2,\dots p}$ basis of $\mathbf{P}^0_p(E_R)$

Properties:

$$\left.\begin{array}{l} \Phi^{E_k,l}_{vertex} \in \mathbf{P}_1(E_R) \\ \Phi^{E_k}_{edge} \in \mathbf{P}^0_p(E_R) \end{array}\right\} \Rightarrow \Phi^{E_k} \in \mathbf{P}_p(E_R)$$

Structure of the mappings (2D, face mapping)

$$\Phi^{F_k}(\xi,\eta) = \sum_{l=0}^{2} \Phi^{F_k,l}_{vertex}(\xi,\eta)$$

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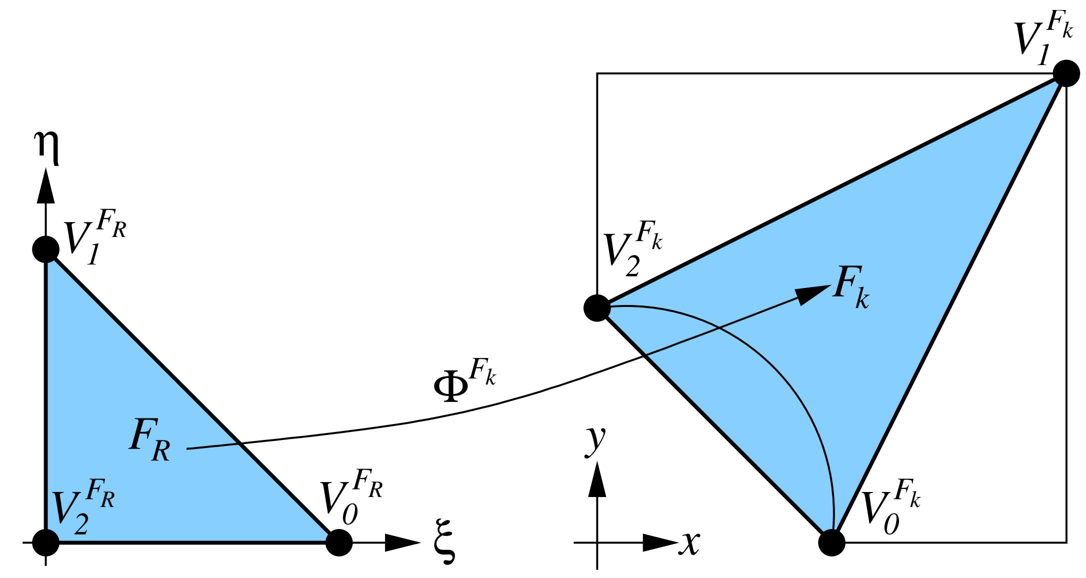

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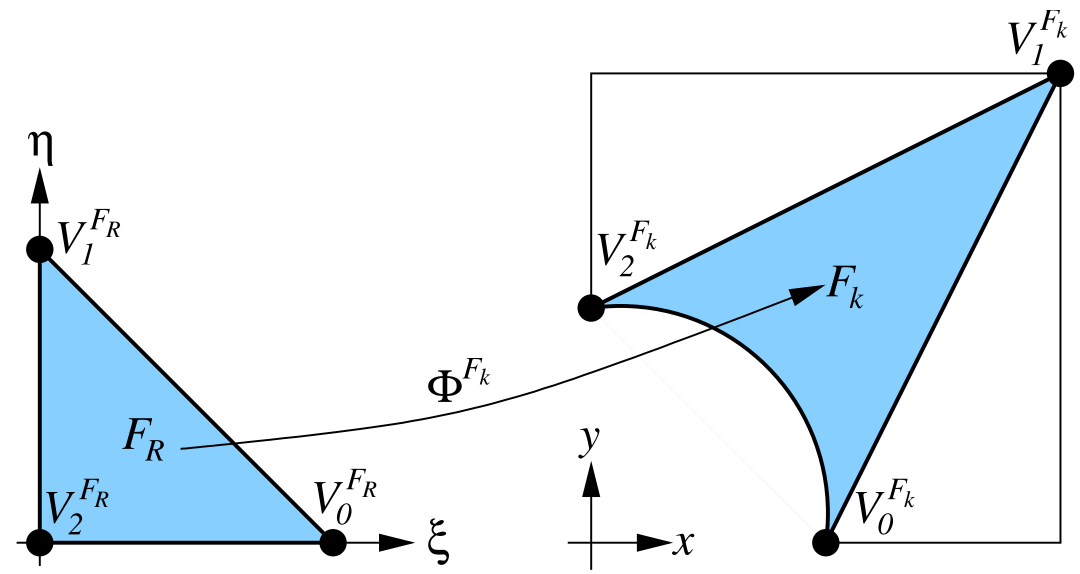
$$\Phi^{F_k,l}_{edge}(\xi,\eta) = \sum_{i=2}^{p} W^{F_k}_{l,i} b_i\left(\frac{\tilde{\lambda}_0}{1-\tilde{\lambda}_2}\right)(1-\tilde{\lambda}_2)^p$$

$\tilde{\lambda}_0$ ... barycentric coordinate representing the first vertex on edge l

$\tilde{\lambda}_2$ ... barycentric coordinate representing vertex $V_l^{F_k}$



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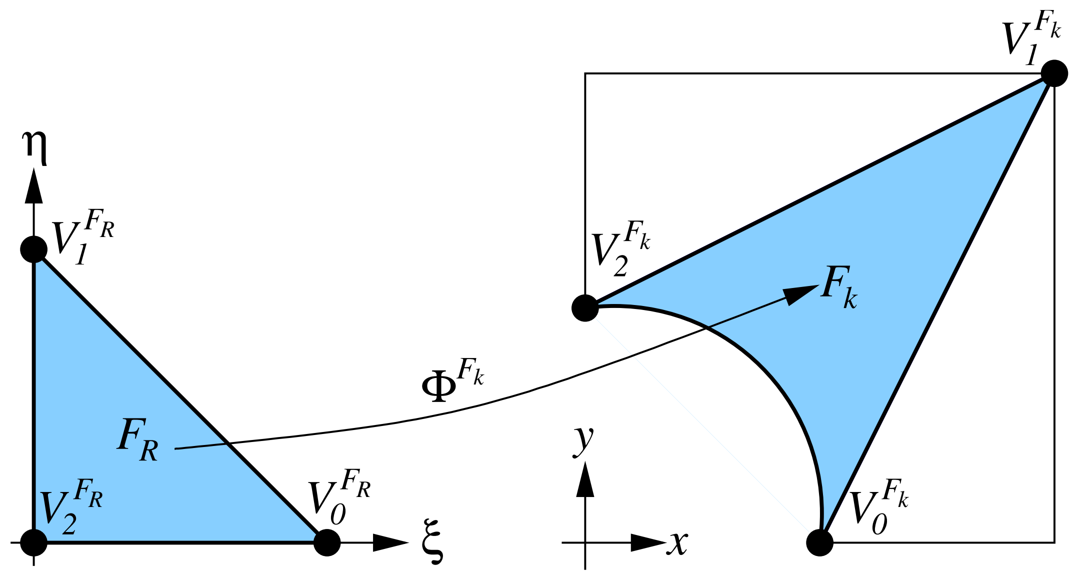

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$\tilde{\lambda}_0$ ... barycentric coordinate representing the first vertex on edge l
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Properties:

$$\left.\begin{array}{l} \Phi^{F_k,l}_{vertex} \in \mathbf{P}_1(F_R) \\ \Phi^{F_k,l}_{edge} \in \mathbf{P}_p(F_R) \text{ and } \Phi^{F_k,l}_{edge} = 0 \text{ on all edges but edge } l \end{array}\right\} \Rightarrow \Phi^{F_k} \in \mathbf{P}_p(F_R)$$

Choosing a suiting basis $\{b_i\}$ of $P_p^0(E_R)$

The basis polynomials appear in the edge-parts of the mappings:

$$\begin{aligned}\Phi_{edge}^{E_k} &= \sum_{i=2}^{p} W_i^{E_k} b_i(\lambda_0) \\ \Phi_{edge}^{F_k,l} &= \sum_{i=2}^{p} W_{l,i}^{F_k} b_i(\frac{\tilde{\lambda}_0}{1-\tilde{\lambda}_2})(1-\tilde{\lambda}_2)^p\end{aligned}$$

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First attempt: Nodal basis

Choose $0 = \xi_1 < \xi_2 < ... < \xi_{p+1} = 1$ (e.g. Gauss-Lobatto points)
$\{b_i\}_{i=2,\dots,p}$ are the unique polynomials $\in P_p^0(E_R)$ fulfilling $b_i(\xi_j) = \delta_{i,j}$

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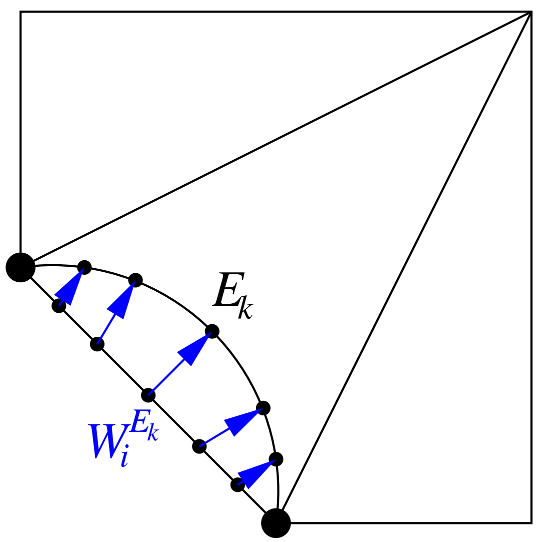


Advantage:

$W_i^{E_k}$, $W_{l,i}^{F_k}$ easy to compute (cf. figure)

Disadvantage:

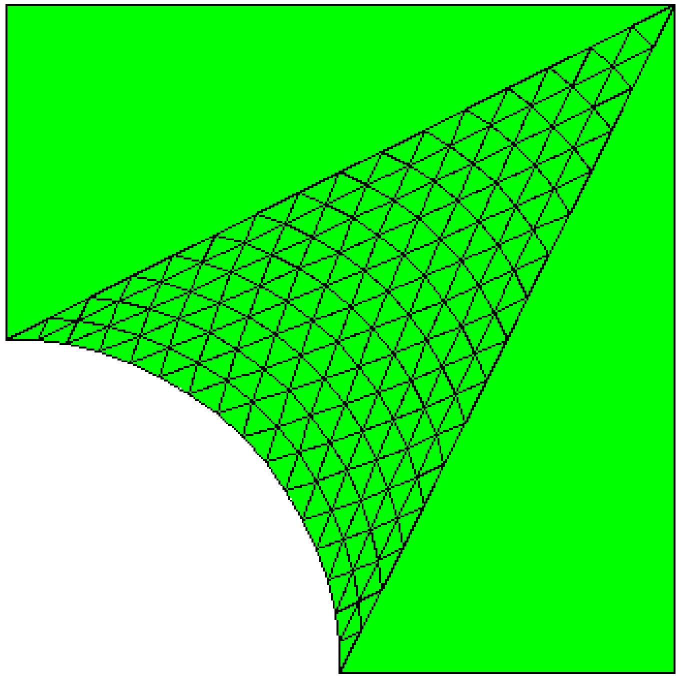
every b_i must be computed separately $\Rightarrow$ slow!

$$\Phi_{edge}^{F_k,l} = \sum_{i=2}^{p} W_{l,i}^{F_k} b_i(\frac{\tilde{\lambda}_0}{1-\tilde{\lambda}_2})(1-\tilde{\lambda}_2)^p$$

Problem:

$(1-\tilde{\lambda}_2)^p$ drops very fast for great p
$\Rightarrow$ curved edge is not extended well into the element

$$\Phi^{F_k,l}_{edge} = \sum_{i=2}^{p} W^{F_k}_{l,i} b_i(\frac{\tilde{\lambda}_0}{1-\tilde{\lambda}_2})(1-\tilde{\lambda}_2)^p$$



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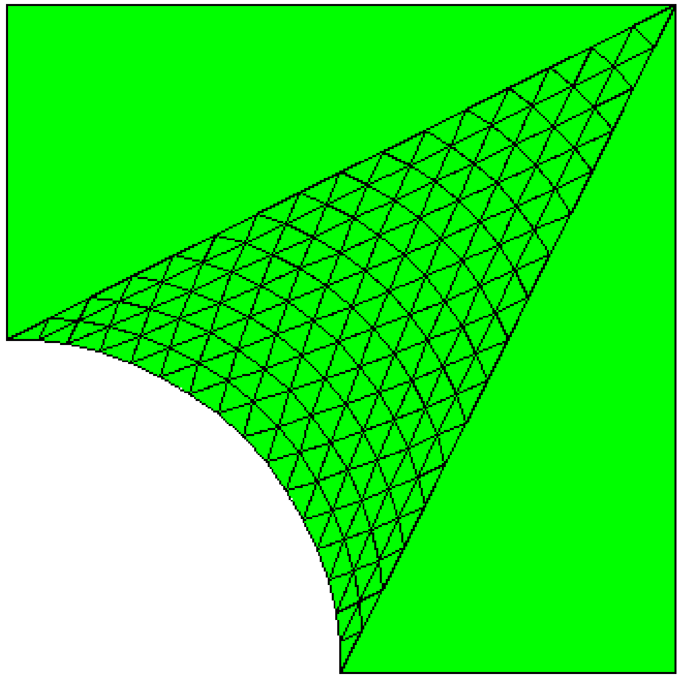

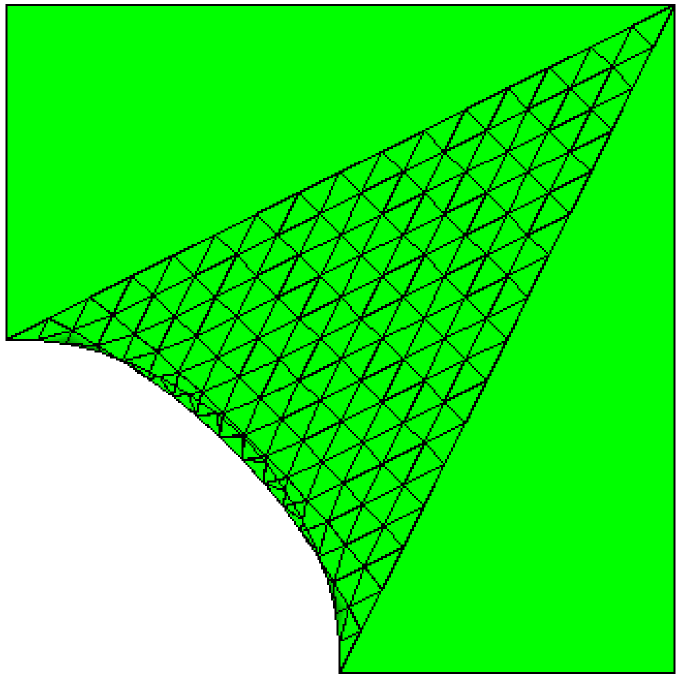

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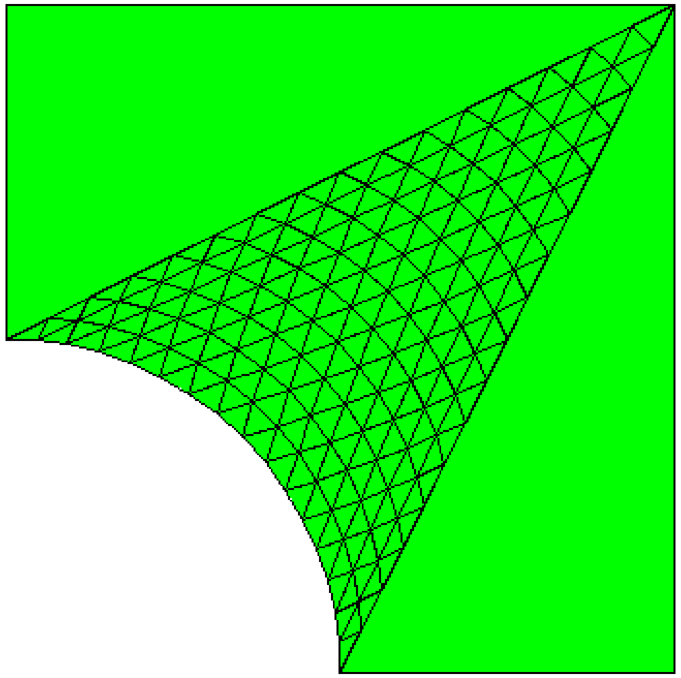

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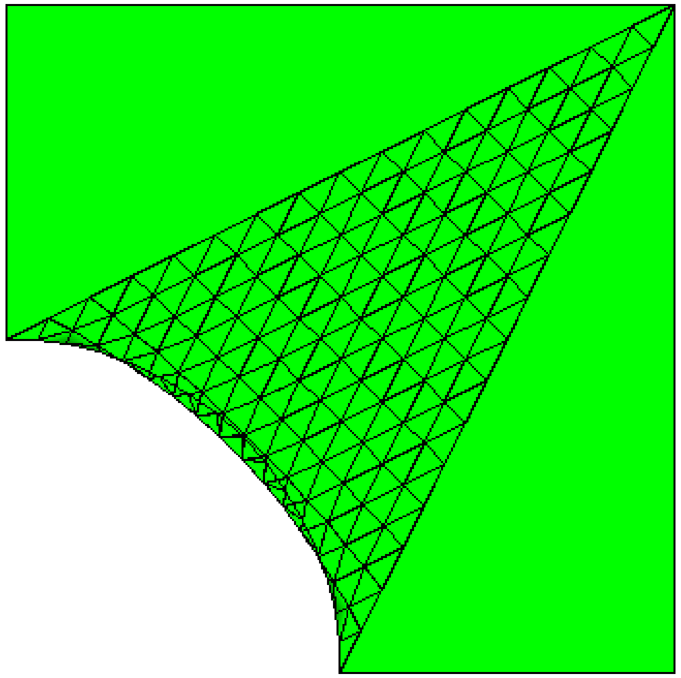


Solution:
Hierarchic basis

$\{b_i\}_{i=2,\ldots,p}$ such, that $b_i \in P_i^0(E_R)$

Advantage:

Extension can be changed to $(1-\tilde{\lambda}_2)^i$
computation of b_i gives all $b_2, \ldots, b_{i-1} \Rightarrow$ fast!

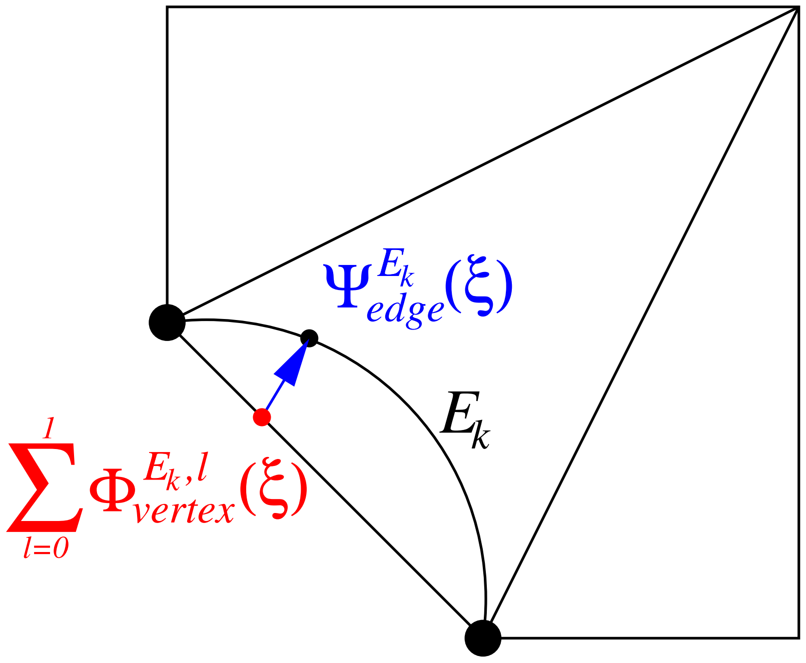
Disadvantage:

$W_i^{E_k}$, $W_{l,i}^{F_k}$ more difficult to compute (but only once!)

How do we get the $W_i^{E_k}$, $W_{l,i}^{F_k}$?

Orthogonal projection onto $P_p^0(E_R)$ in $H^1(E_R)$-seminorm
Find $\Phi_{edge}^{E_k} \in P_p^0(E_R)$ such that

$$|\Phi_{edge}^{E_k} - \Psi_{edge}^{E_k}|_{H^1(E_R)} = \min_{\Phi \in P_p^0(E_R)} |\Phi - \Psi_{edge}^{E_k}|_{H^1(E_R)}$$



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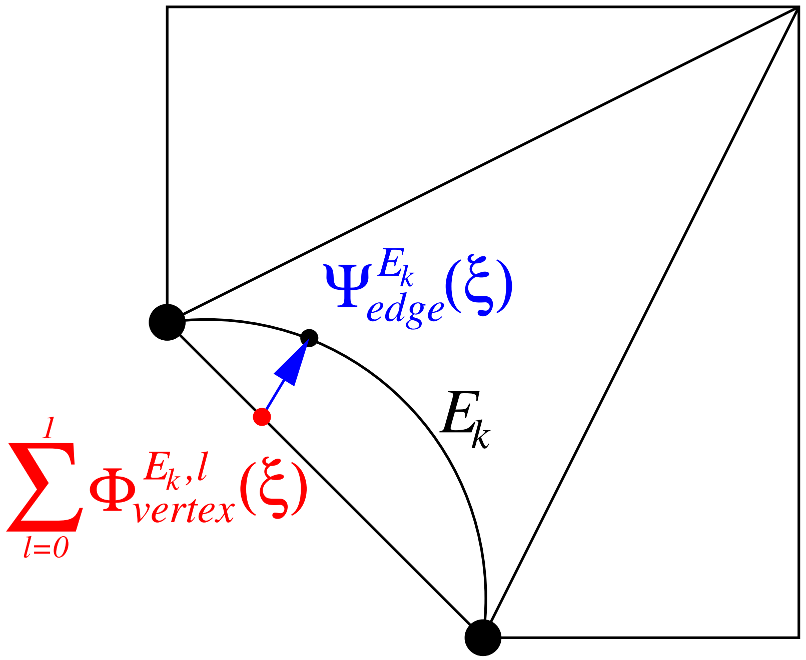

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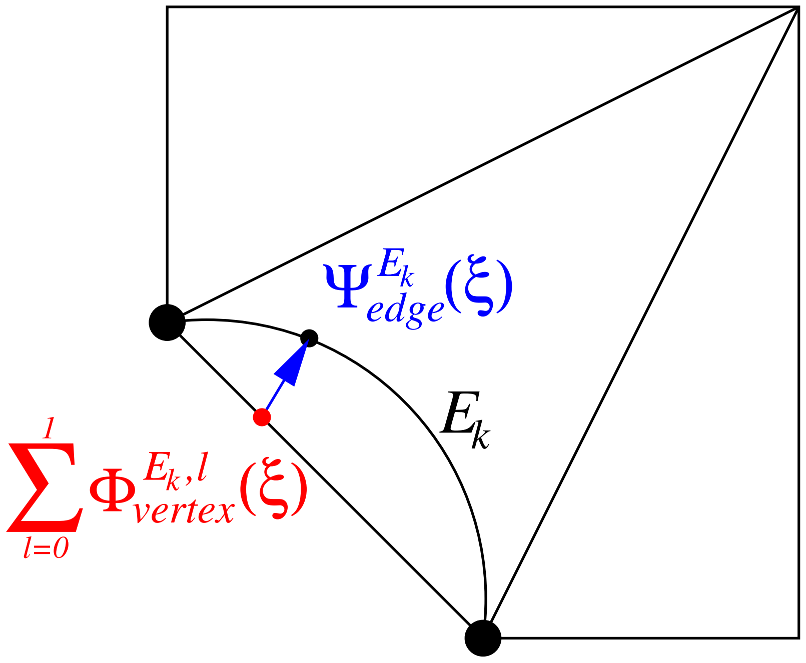

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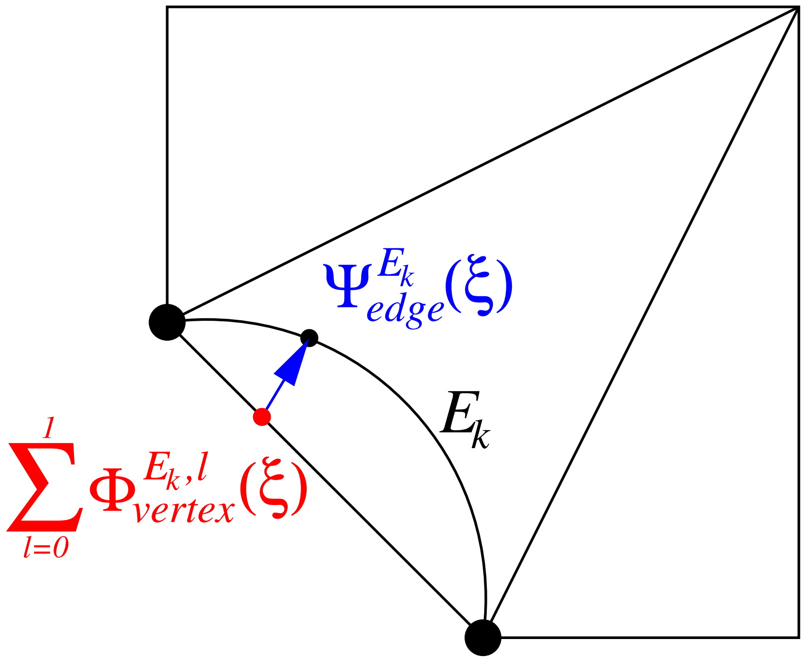
$$\langle \sum_{i=2}^{p} W_i^{E_k} b_i, b_j \rangle = \langle \Psi_{edge}^{E_k}, b_j \rangle$$



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Orthogonal projection onto $P_p^0(E_R)$ in $H^1(E_R)$-seminorm

Find $\Phi_{edge}^{E_k} \in P_p^0(E_R)$ such that

$$|\Phi_{edge}^{E_k} - \Psi_{edge}^{E_k}|_{H^1(E_R)} = \min_{\Phi \in P_p^0(E_R)} |\Phi - \Psi_{edge}^{E_k}|_{H^1(E_R)}$$

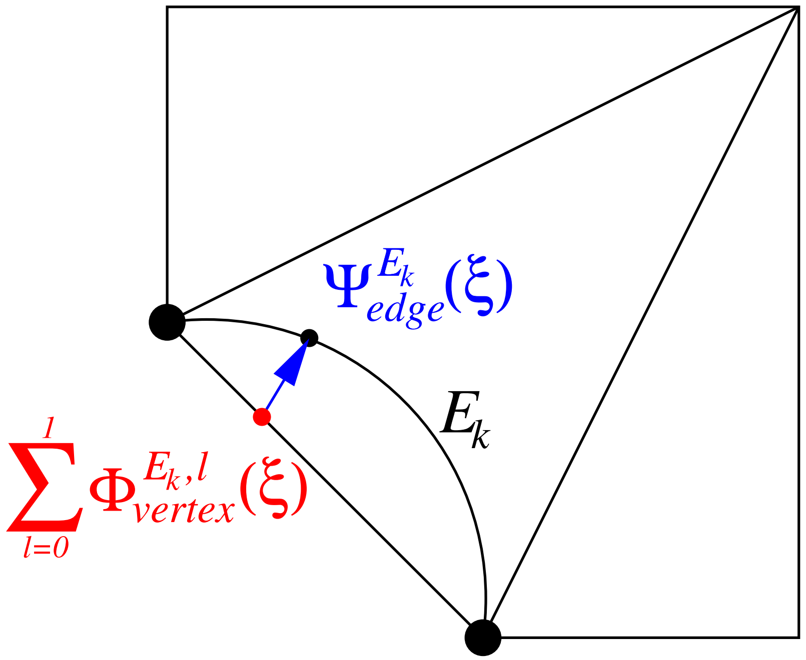
$$\Longleftrightarrow$$

$$\forall \Phi \in P_p^0(E_R): \quad \langle \Phi_{edge}^{E_k}, \Phi \rangle = \langle \Psi_{edge}^{E_k}, \Phi \rangle$$

$$\forall b_j: \quad \langle \Phi_{edge}^{E_k}, b_j \rangle = \langle \Psi_{edge}^{E_k}, b_j \rangle$$

$$\langle \sum_{i=2}^{p} W_i^{E_k} b_i, b_j \rangle = \langle \Psi_{edge}^{E_k}, b_j \rangle$$

$$\sum_{i=2}^{p} W_i^{E_k} \int_{E_R} b_i'(\xi) b_j'(\xi)\, d\xi = \int_{E_R} \left(\Psi_{edge}^{E_k}\right)'(\xi) b_j'(\xi)\, d\xi$$



How do we get the $W_i^{E_k}$, $W_{l,i}^{F_k}$?

Orthogonal projection onto $P_p^0(E_R)$ in $H^1(E_R)$-seminorm
Find $\Phi_{edge}^{E_k} \in P_p^0(E_R)$ such that

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$$\Longleftrightarrow$$

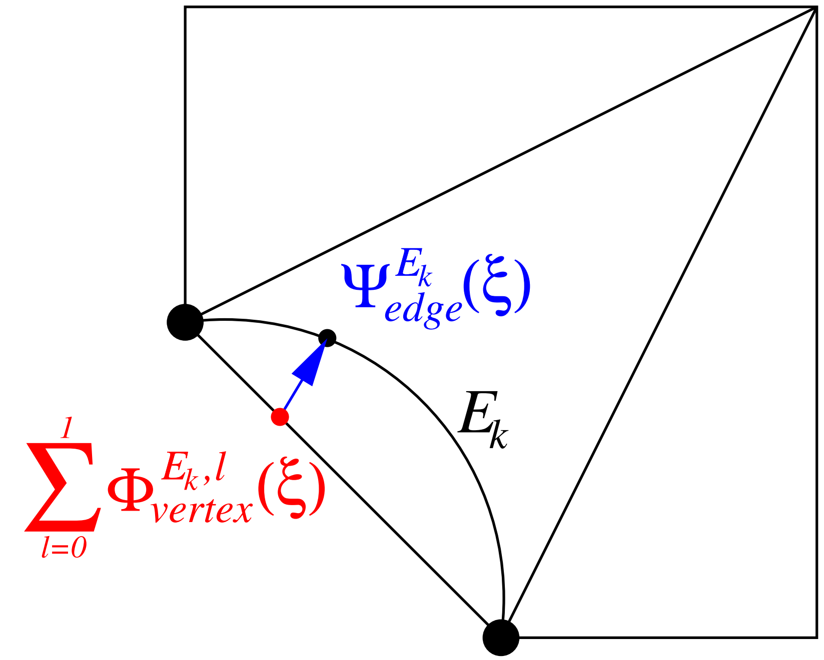
$$\forall \Phi \in P_p^0(E_R): \quad \langle \Phi_{edge}^{E_k}, \Phi \rangle = \langle \Psi_{edge}^{E_k}, \Phi \rangle$$

$$\forall b_j: \quad \langle \Phi_{edge}^{E_k}, b_j \rangle = \langle \Psi_{edge}^{E_k}, b_j \rangle$$

$$\langle \sum_{i=2}^{p} W_i^{E_k} b_i, b_j \rangle = \langle \Psi_{edge}^{E_k}, b_j \rangle$$

$$\sum_{i=2}^{p} W_i^{E_k} \int_{E_R} b_i'(\xi) b_j'(\xi)\, d\xi = \int_{E_R} \left(\Psi_{edge}^{E_k}\right)'(\xi) b_j'(\xi)\, d\xi$$

$$\sum_{i=2}^{p} W_i^{E_k} \underbrace{\int_{E_R} b_i'(\xi) b_j'(\xi)\, d\xi}_{=:A_{ij}} = \underbrace{-\int_{E_R} \Psi_{edge}^{E_k}(\xi) b_j''(\xi)\, d\xi}_{=:f_j} + \underbrace{\Psi_{edge}^{E_k}(1) b_j'(1)}_{=0} - \underbrace{\Psi_{edge}^{E_k}(0) b_j'(0)}_{=0}$$



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$$\Longleftrightarrow$$

$$\forall \Phi \in P_p^0(E_R): \quad \langle \Phi_{edge}^{E_k}, \Phi \rangle = \langle \Psi_{edge}^{E_k}, \Phi \rangle$$

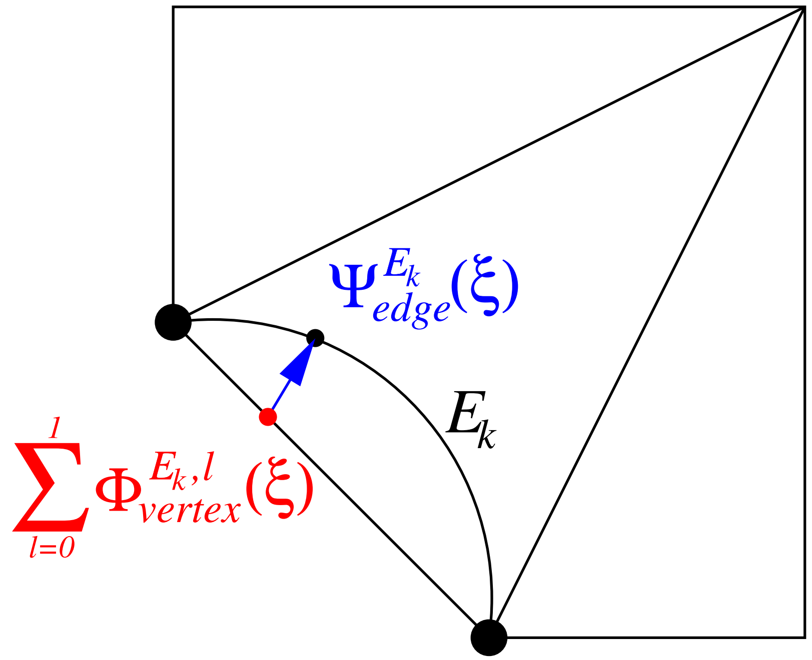
$$\forall b_j: \quad \langle \Phi_{edge}^{E_k}, b_j \rangle = \langle \Psi_{edge}^{E_k}, b_j \rangle$$

$$\langle \sum_{i=2}^{p} W_i^{E_k} b_i, b_j \rangle = \langle \Psi_{edge}^{E_k}, b_j \rangle$$

$$\sum_{i=2}^{p} W_i^{E_k} \int_{E_R} b_i'(\xi) b_j'(\xi)\, d\xi = \int_{E_R} \left(\Psi_{edge}^{E_k}\right)' (\xi) b_j'(\xi)\, d\xi$$

$$\sum_{i=2}^{p} W_i^{E_k} \underbrace{\int_{E_R} b_i'(\xi) b_j'(\xi)\, d\xi}_{=:A_{ij}} = \underbrace{-\int_{E_R} \Psi_{edge}^{E_k}(\xi) b_j''(\xi)\, d\xi}_{=:f_j} + \underbrace{\Psi_{edge}^{E_k}(1) b_j'(1)}_{=0} - \underbrace{\Psi_{edge}^{E_k}(0) b_j'(0)}_{=0}$$

$$W^{E_k} = A^{-1} f$$



How to get the derivatives: Edge mapping

$$\Phi^{E_k}(\xi) = \sum_{l=0}^{1} \Phi_{vertex}^{E_k,l}(\xi) + \Phi_{edge}^{E_k}(\xi)$$

with

$$\begin{aligned} \Phi_{vertex}^{E_k,l}(\xi) &= \lambda_l V_l^{E_k} \\ \Phi_{edge}^{E_k}(\xi) &= \sum_{i=2}^{p} W_i^{E_k} b_i(\lambda_0) \end{aligned}$$

gives

$$\frac{d\Phi^{E_k}}{d\xi}(\xi) = \sum_{l=0}^{1} \frac{d\Phi_{vertex}^{E_k,l}}{d\xi}(\xi) + \frac{d\Phi_{edge}^{E_k}}{d\xi}(\xi)$$

with

$$\begin{aligned} \frac{d\Phi_{vertex}^{E_k,l}}{d\xi}(\xi) &= \frac{d\lambda_l}{d\xi}(\xi) V_l^{E_k} \\ \frac{d\Phi_{edge}^{E_k}}{d\xi}(\xi) &= \sum_{i=2}^{p} W_i^{E_k} b_i'(\lambda_0) \frac{d\lambda_0}{d\xi}(\xi) \end{aligned}$$

How to get the derivatives: Face mapping

Similarly we get

$$\frac{\partial \Phi^{F_k}(\xi,\eta)}{\partial \xi} = \sum_{l=0}^{2} \frac{\partial \Phi^{F_k,l}_{vertex}}{\partial \xi}(\xi,\eta) + \sum_{l=0}^{2} \frac{\partial \Phi^{F_k,l}_{edge}(\xi,\eta)}{\partial \xi}$$

with

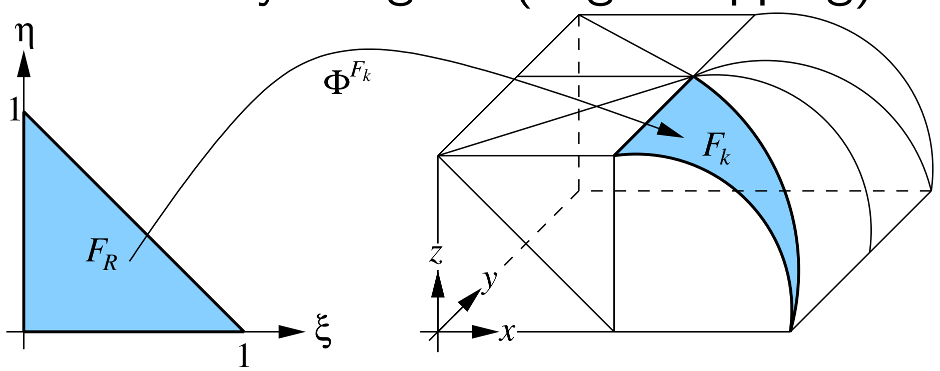
$$\begin{aligned}
\frac{\partial \Phi^{F_k,l}_{vertex}}{\partial \xi}(\xi,\eta) &= \frac{\partial \lambda_l}{\partial \xi}(\xi,\eta) V_l^{F_k} \\
\frac{\partial \Phi^{F_k,l}_{edge}}{\partial \xi}(\xi,\eta) &= \sum_{i=2}^{p} W^{F_k}_{l,i} \left[b_i'(\frac{\tilde\lambda_0}{1-\tilde\lambda_2}) \frac{\frac{\partial \tilde\lambda_0}{\partial \xi}(1-\tilde\lambda_2) - \tilde\lambda_0(-\frac{\partial \tilde\lambda_2}{\partial \xi})}{(1-\tilde\lambda_2)^2} (1-\tilde\lambda_2)^i + i b_i(\frac{\tilde\lambda_0}{1-\tilde\lambda_2})(1-\tilde\lambda_2)^{i-1}(-\frac{\partial \tilde\lambda_2}{\partial \xi}) \right]
\end{aligned}$$

and analogous

$$\frac{\partial \Phi^{F_k}(\xi,\eta)}{\partial \eta}$$

3D: What do we need?

Boundary integrals (edge mapping)

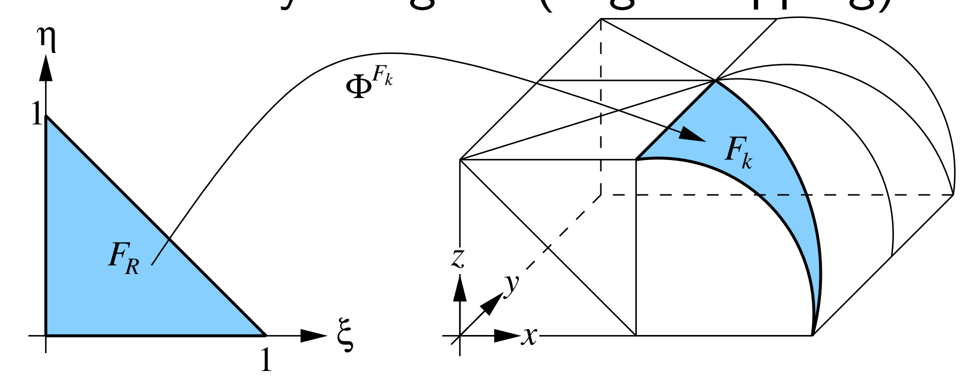


$$\int_{F_k} f(x) \cdot dx = \int_{F_R} f(\Phi^{F_k}(\xi)) \left| \frac{\partial \Phi^{F_k}}{\partial \xi} \right| \cdot d\xi$$

$$\begin{aligned} \xi &= (\xi, \eta) \\ x &= (x, y, z) \\ x &= \Phi^{F_k}(\xi) \end{aligned}$$

3D: What do we need?

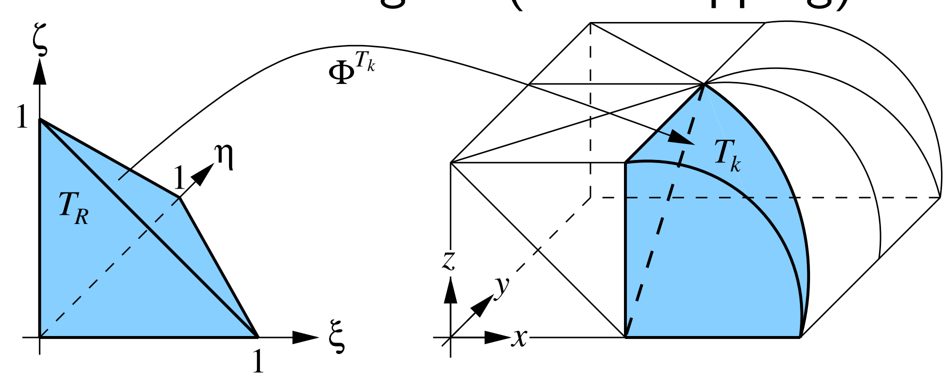
Boundary integrals (edge mapping)



$$\int_{F_k} f(x) \cdot dx = \int_{F_R} f(\Phi^{F_k}(\xi)) \left| \frac{\partial \Phi^{F_k}}{\partial \xi} \right| \cdot d\xi$$

$$\begin{aligned} \xi &= (\xi, \eta) \\ x &= (x, y, z) \\ x &= \Phi^{F_k}(\xi) \end{aligned}$$

Volume integrals (face mapping)

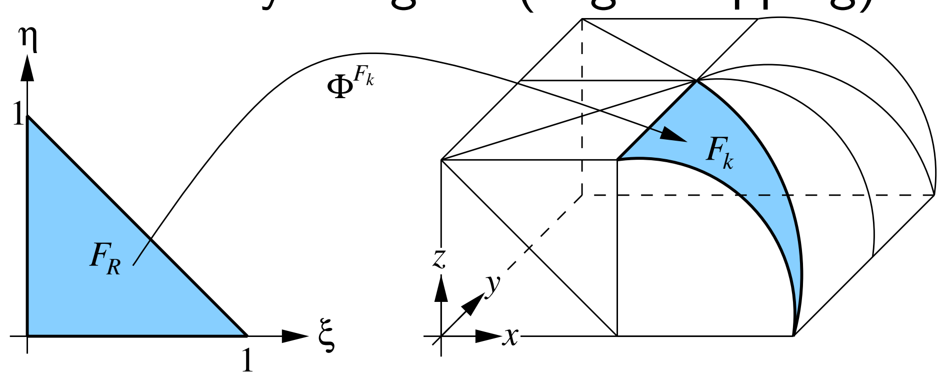


$$\int_{T_k} f(x)\, dx = \int_{T_R} f(\Phi^{T_k}(\xi)) \left| \frac{\partial \Phi^{T_k}}{\partial \xi} \right| d\xi$$

$$\begin{aligned} \xi &= (\xi, \eta, \zeta) \\ x &= (x, y, z) \\ x &= \Phi^{T_k}(\xi) \end{aligned}$$

3D: What do we need?

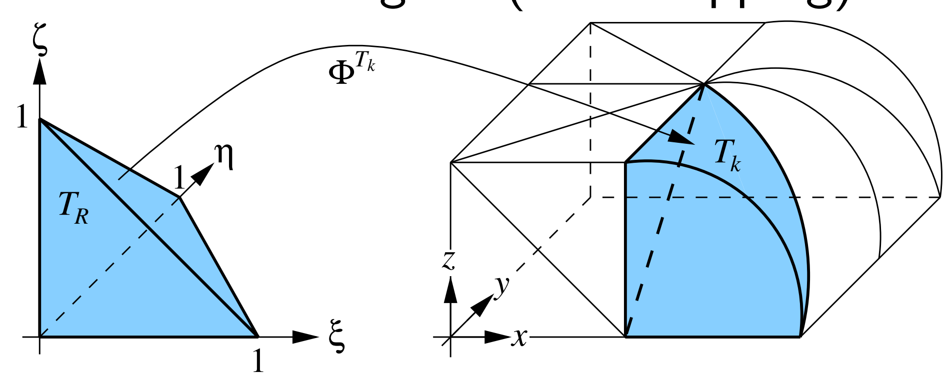
Boundary integrals (edge mapping)



$$\int_{F_k} f(x) \cdot dx = \int_{F_R} f(\Phi^{F_k}(\xi)) \left| \frac{\partial \Phi^{F_k}}{\partial \xi} \right| \cdot d\xi$$

$$\begin{aligned} \xi &= (\xi, \eta) \\ x &= (x, y, z) \\ x &= \Phi^{F_k}(\xi) \end{aligned}$$

Volume integrals (face mapping)



$$\int_{T_k} f(x)\, dx = \int_{T_R} f(\Phi^{T_k}(\xi)) \left| \frac{\partial \Phi^{T_k}}{\partial \xi} \right| d\xi$$

$$\begin{aligned} \xi &= (\xi, \eta, \zeta) \\ x &= (x, y, z) \\ x &= \Phi^{T_k}(\xi) \end{aligned}$$

Construct:

$$\Phi^{F_k}, \frac{\partial \Phi^{F_k}}{\partial \xi}$$

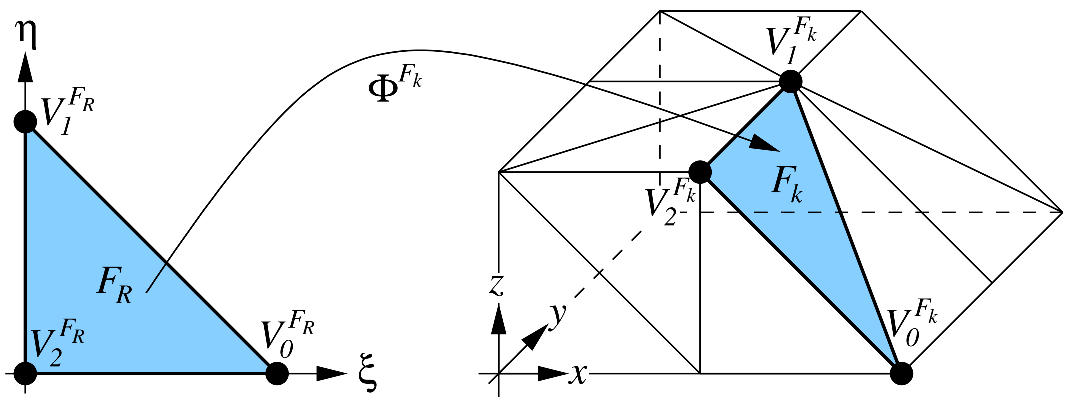
$$\Phi^{T_k}, \frac{\partial \Phi^{T_k}}{\partial \xi}$$

Structure of the mappings (3D, face mapping)

$$\Phi^{F_k}(\xi,\eta) = \sum_{l=0}^{2} \Phi^{F_k,l}_{vertex}(\xi,\eta)$$

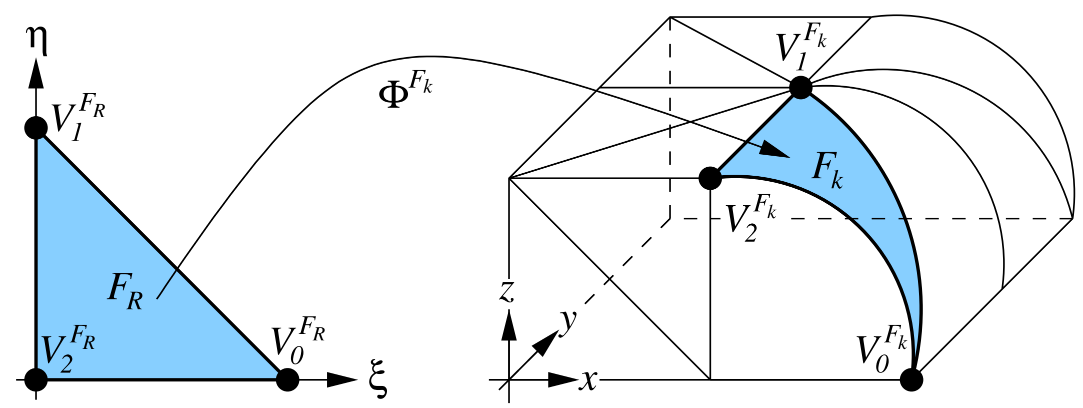
with

$$\Phi^{F_k,l}_{vertex}(\xi,\eta) \quad = \quad \lambda_l V_l^{F_k}$$



Structure of the mappings (3D, face mapping)

$$\Phi^{F_k}(\xi,\eta) = \sum_{l=0}^{2} \Phi^{F_k,l}_{vertex}(\xi,\eta) + \sum_{l=0}^{2} \Phi^{F_k,l}_{edge}(\xi,\eta) + \Phi^{F_k}_{face}(\xi,\eta)$$



with

$$\begin{aligned}
\Phi^{F_k,l}_{vertex}(\xi,\eta) &= \lambda_l V_l^{F_k} \\
\Phi^{F_k,l}_{edge}(\xi,\eta) &= \sum_{i=2}^{p} W^{F_k}_{l,i} b_i\left(\frac{\tilde{\lambda}_0}{1-\tilde{\lambda}_2}\right)(1-\tilde{\lambda}_2)^i \\
\Phi^{F_k}_{face}(\xi,\eta) &= \sum_{\substack{i,j\geq 2\\ i+j\leq p+1}} U^{F_k}_{i,j} b_i(\lambda_2) b_j\left(\frac{\lambda_0}{1-\lambda_2}\right)(1-\lambda_2)^{j-1}
\end{aligned}$$

and

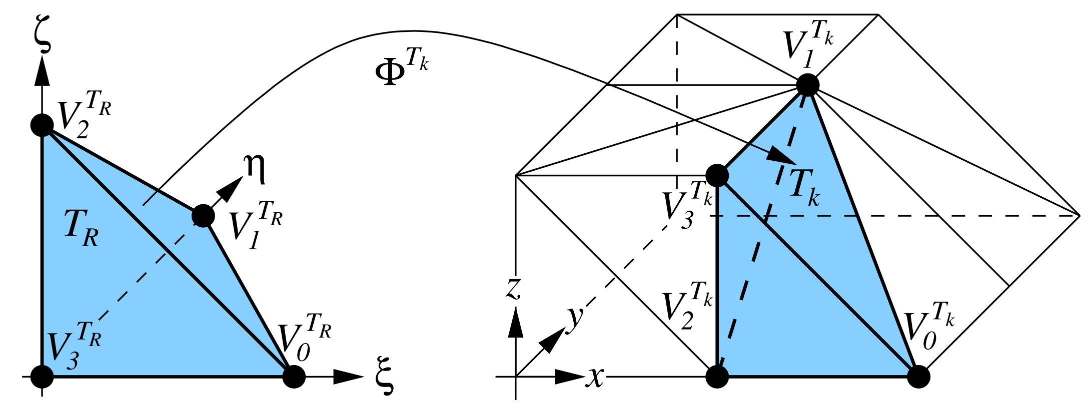
$$W^{F_k}_{l,i},\, U^{F_k}_{i,j} \in \mathbf{R}^3$$

Properties:

$$\left.\begin{array}{l} \Phi^{F_k,l}_{vertex} \in \mathbf{P}_1(F_R) \\ \Phi^{F_k,l}_{edge} \in \mathbf{P}_p(F_R) \text{ and } \Phi^{F_k,l}_{edge} = 0 \text{ on all edges but edge } l \\ \Phi^{F_k}_{face} \in \mathbf{P}^0_p(F_R) \end{array}\right\} \Rightarrow \Phi^{F_k} \in \mathbf{P}_p(F_R)$$

Structure of the mappings (3D, tetraeder mapping)

$$\Phi^{T_k}(\xi,\eta,\zeta) \quad = \quad \sum_{l=0}^{3} \Phi^{T_k,l}_{vertex}(\xi,\eta,\zeta)$$

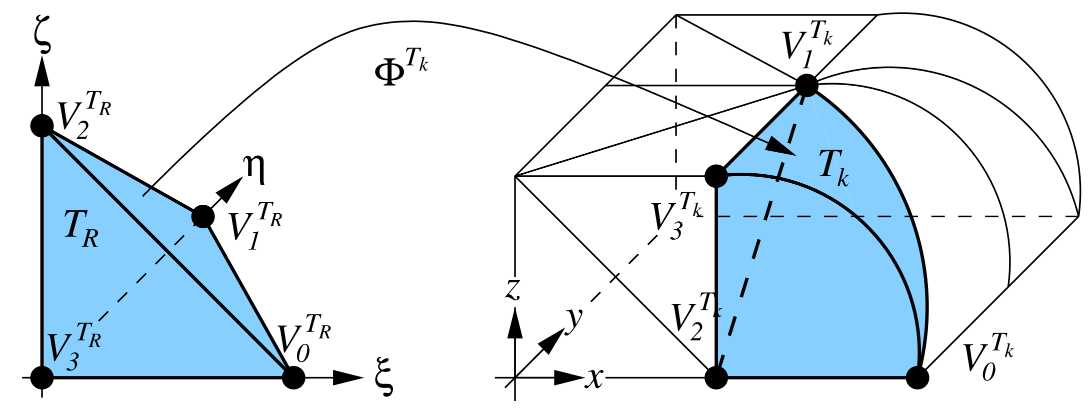


with

$$\Phi^{T_k,l}_{vertex}(\xi,\eta,\zeta) \quad = \quad \lambda_l V_l^{E_k}$$

Structure of the mappings (3D, tetraeder mapping)

$$\Phi^{T_k}(\xi,\eta,\zeta) = \sum_{l=0}^{3} \Phi^{T_k,l}_{vertex}(\xi,\eta,\zeta) + \sum_{l=0}^{5} \Phi^{T_k,l}_{edge}(\xi,\eta,\zeta) + \sum_{l=0}^{3} \Phi^{T_k,l}_{face}(\xi,\eta,\zeta)$$



with

$$\Phi^{T_k,l}_{vertex}(\xi,\eta,\zeta) = \lambda_l V_l^{E_k}$$

$$\Phi^{T_k,l}_{edge}(\xi,\eta,\zeta) = \sum_{i=2}^{p} W^{F_k}_{l,i} b_i\left(\frac{\tilde\lambda_0}{1-\tilde\lambda_2-\tilde\lambda_3}\right)(1-\tilde\lambda_2-\tilde\lambda_3)^i$$

$$\Phi^{T_k,l}_{face}(\xi,\eta,\zeta) = \sum_{\substack{i,j\geq 2\\ i+j\leq p+1}} U^{T_k}_{l,i,j} b_i\left(\frac{\tilde\lambda_0}{1-\tilde\lambda_2-\tilde\lambda_3}\right) b_j\left(\frac{\tilde\lambda_2}{1-\tilde\lambda_3}\right)(1-\tilde\lambda_2-\tilde\lambda_3)^{i-1}(1-\tilde\lambda_3)^j$$

and

$$W^{T_k}_{l,i,j},\, U^{T_k}_{l,i,j} \in \mathbf{R}^3$$

Properties:

$$\left.\begin{array}{l} \Phi_{vertex}^{T_k,l} \in \mathbf{P}_1(T_R) \\ \Phi_{edge}^{T_k,l} \in \mathbf{P}_p(T_R) \text{ and } \Phi_{edge}^{T_k,l} = 0 \text{ on all edges but edge } l \\ \Phi_{face}^{T_k,l} \in \mathbf{P}_p(T_R) \text{ and } \Phi_{face}^{T_k,l} = 0 \text{ on all faces but face } l \end{array}\right\} \Rightarrow \Phi^{T_k} \in \mathbf{P}_p(T_R)$$

How do we get the $W_{l,i}^{F_k}$, $U_{i,j}^{F_k}$?

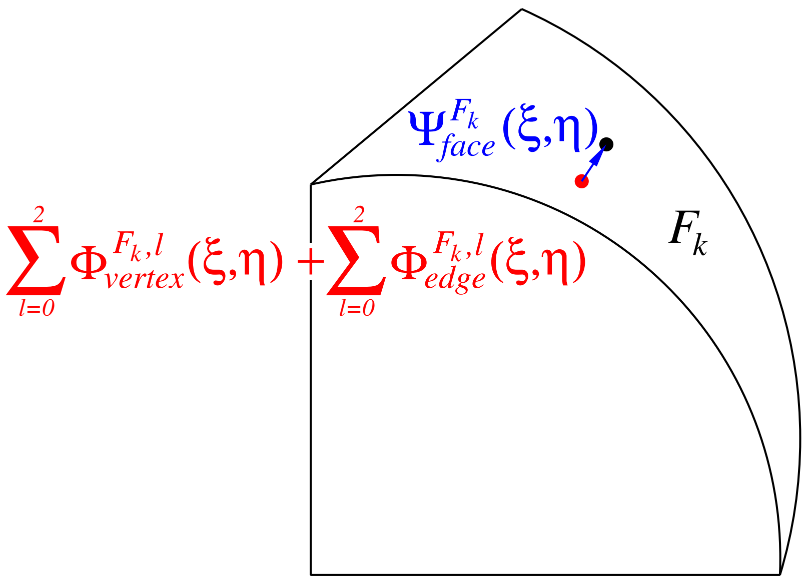
$W_{l,i}^{F_k}$ are computed like in 2D

How do we get the $W_{l,i}^{F_k}$, $U_{i,j}^{F_k}$?

$W_{l,i}^{F_k}$ are computed like in 2D

$U_{i,j}^{F_k}$ are found through orthogonal projection onto $P_p^0(F_R)$ in $H^1(F_R)$-seminorm
Find $\Phi_{face}^{F_k} \in P_p^0(F_R)$ such that

$$|\Phi_{face}^{F_k} - \Psi_{face}^{F_k}|_{H^1(F_R)} = \min_{\Phi \in P_p^0(F_R)} |\Phi - \Psi_{face}^{F_k}|_{H^1(F_R)}$$

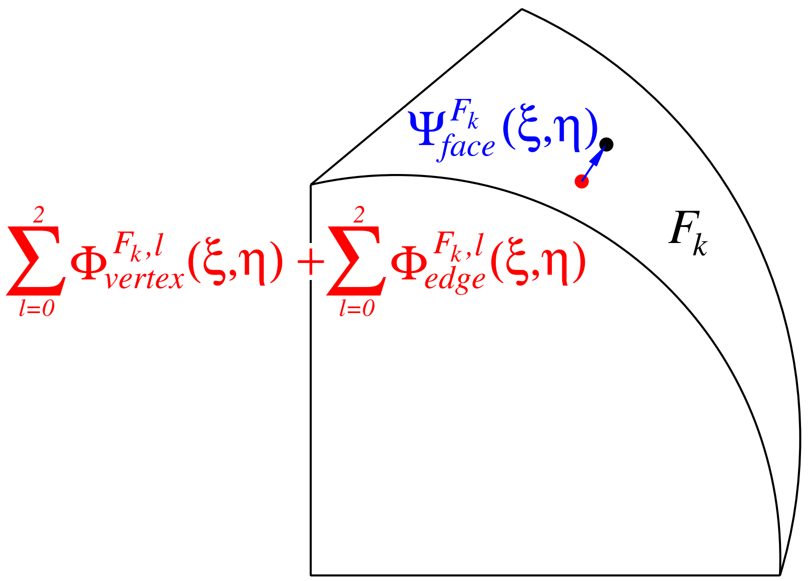


How do we get the $W^{F_k}_{l,i}$, $U^{F_k}_{i,j}$?

$W^{F_k}_{l,i}$ are computed like in 2D

$U^{F_k}_{i,j}$ are found through orthogonal projection onto $P^0_p(F_R)$ in $H^1(F_R)$-seminorm
Find $\Phi^{F_k}_{face} \in P^0_p(F_R)$ such that

$$|\Phi^{F_k}_{face} - \Psi^{F_k}_{face}|_{H^1(F_R)} = \min_{\Phi \in P^0_p(F_R)} |\Phi - \Psi^{F_k}_{face}|_{H^1(F_R)}$$

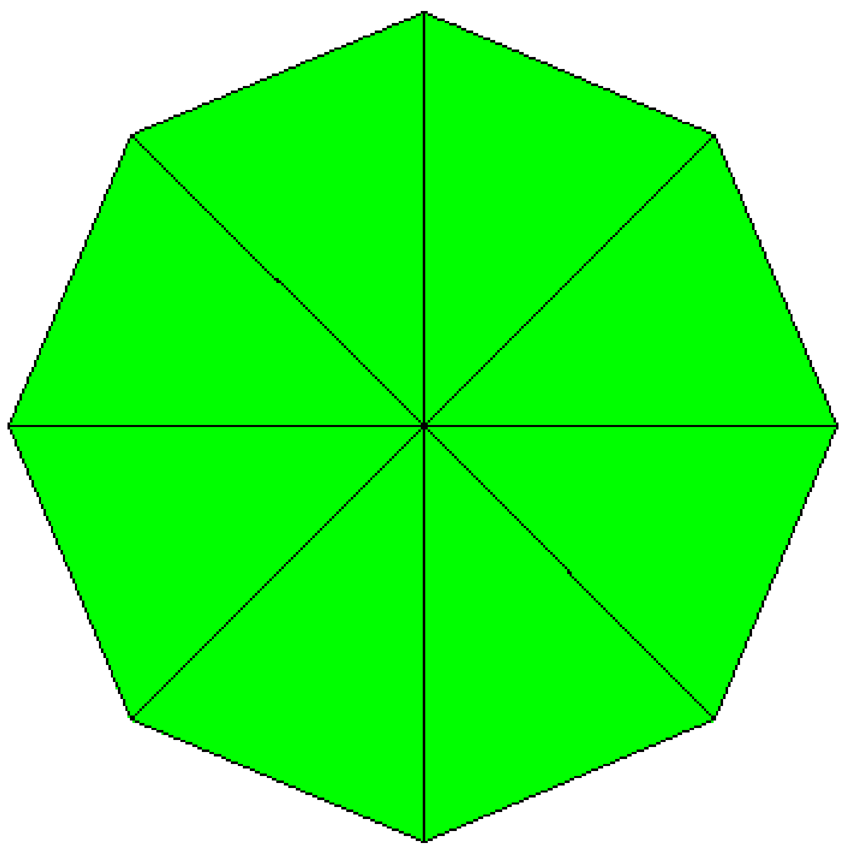


Utilizing that $\{b_i(\xi) b_j(\frac{\eta}{1-\xi})(1-\xi)^{j-1}\}_{i,j\geq 2,\, i+j \leq p+1}$ is a basis of $P^0_p(F_R)$, and rewriting it $\{B_i\}$, it follows

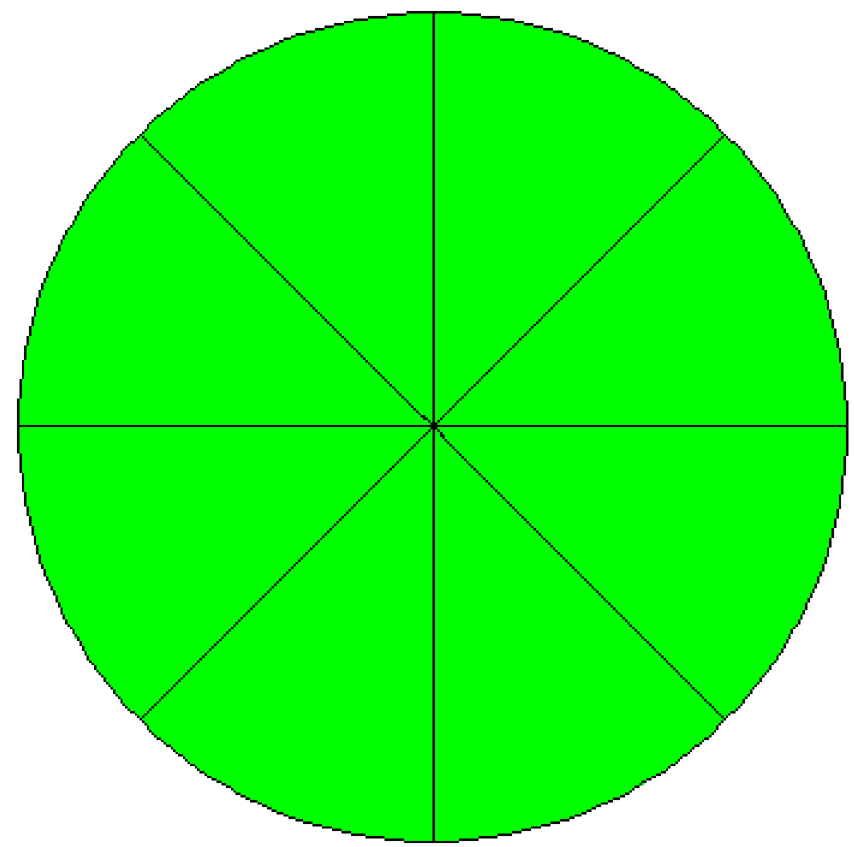
$$\sum_i U^{F_k}_{i,j} \underbrace{\int_{F_R} \nabla B_i \nabla B_j \, d\xi}_{=:A_{ij}} = \underbrace{-\int_{F_R} \Psi^{F_k}_{edge} \Delta B_j \, d\xi + \int_{\partial F_R} \Psi^{F_k}_{edge} \frac{\partial B_j}{\partial n} d\xi}_{=:f_j}$$

$$U^{F_k} = A^{-1} f$$

Some pictures: Circle

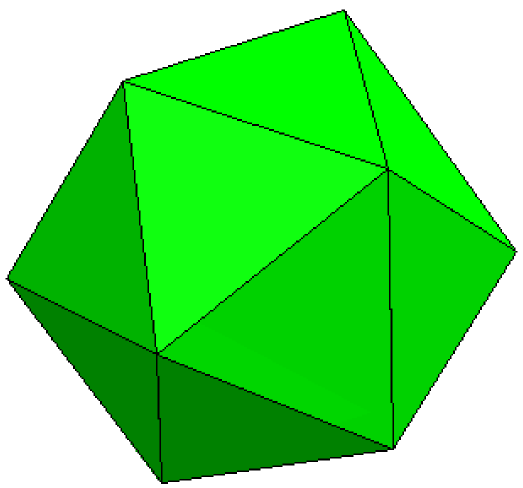


$p = 1$

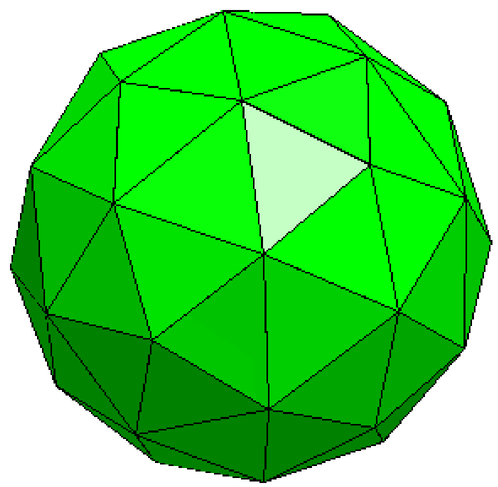


$p = 5$

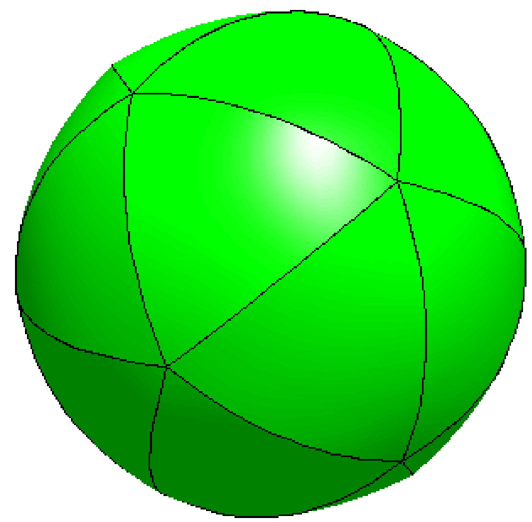
Some pictures: Sphere



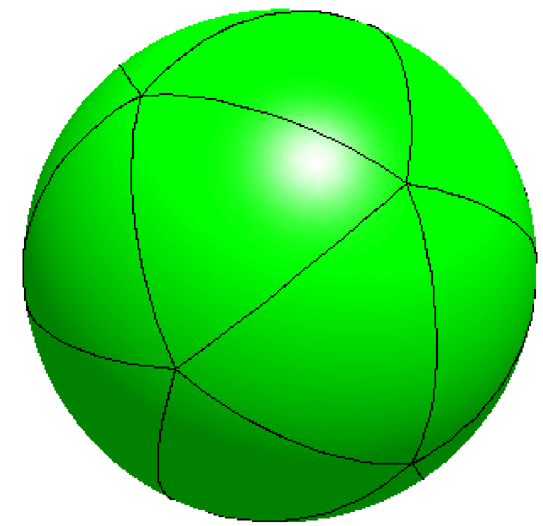
$p = 1,\ N_{SE} = 20$



$p = 1,\ N_{SE} = 80$

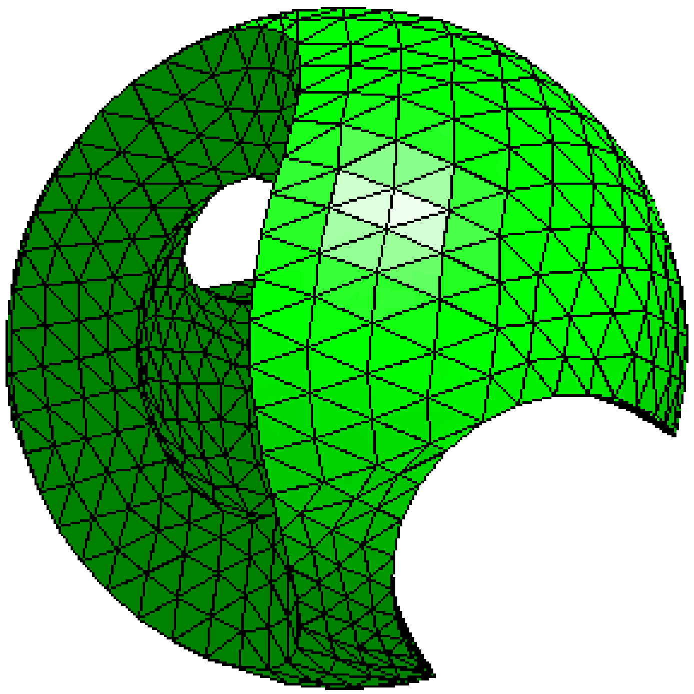


$p = 2,\ N_{SE} = 20$

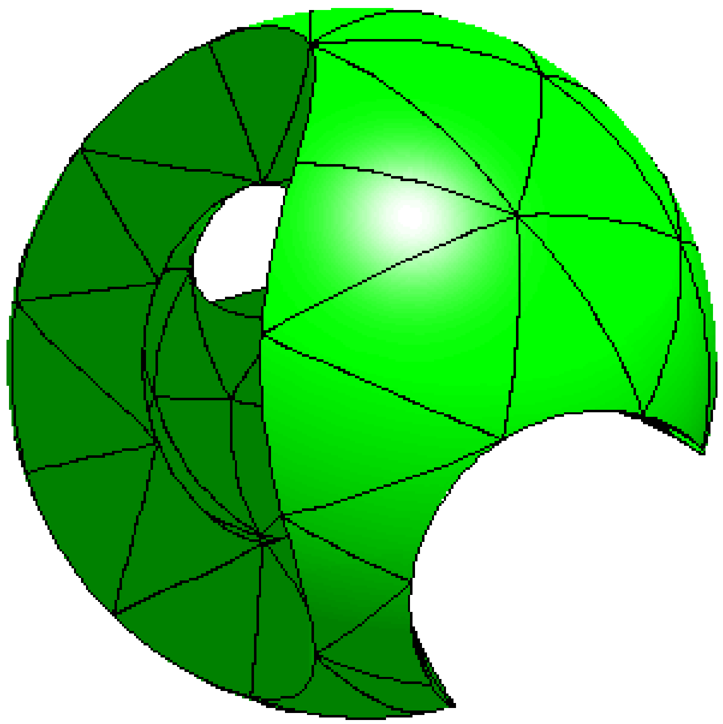


$p = 5,\ N_{SE} = 20$

Some pictures: Sculpture

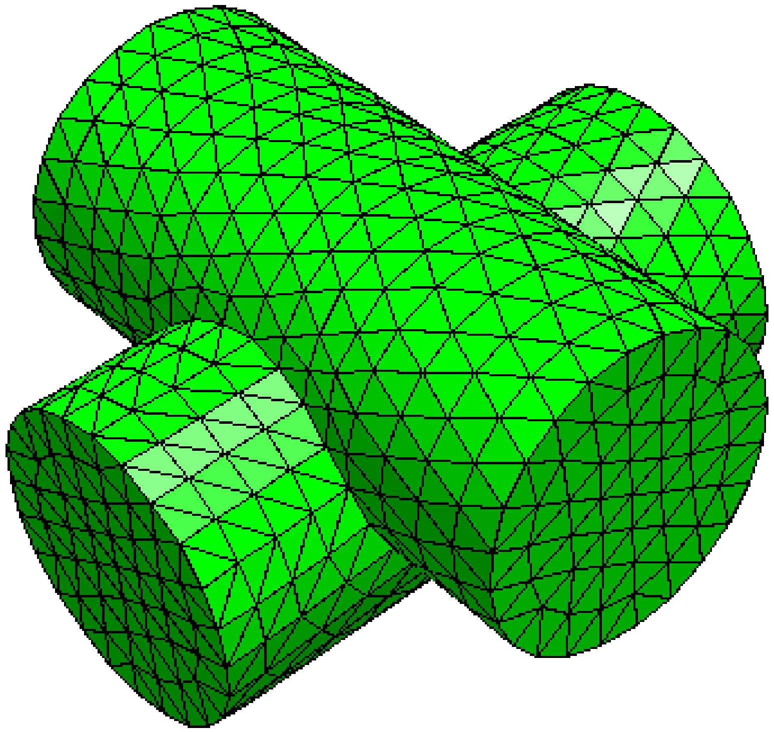


$p = 1,\ N_{SE} = 2112$

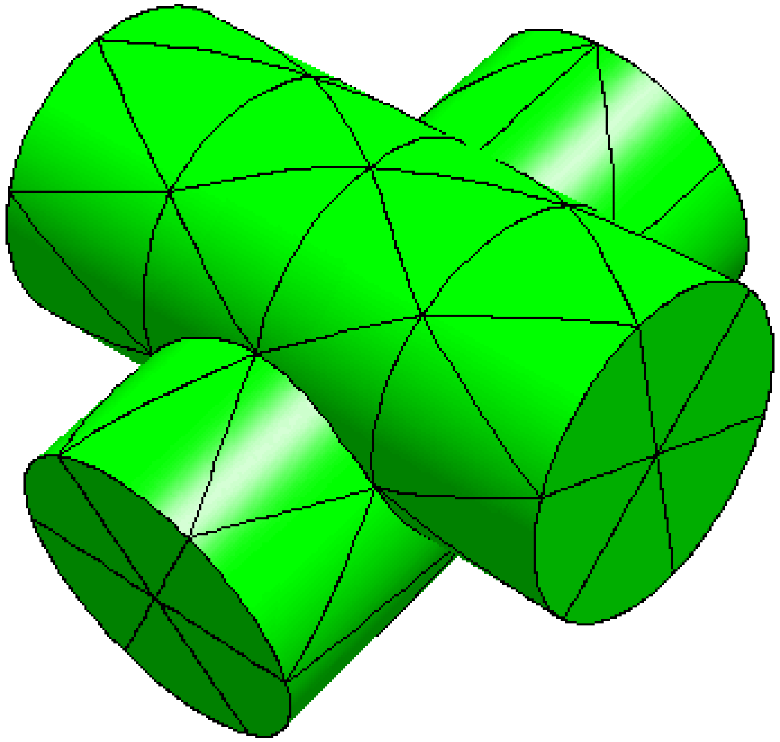


$p = 5,\ N_{SE} = 132$

Some pictures: Two cylinders



$p = 1,\ N_{SE} = 1505$



$p = 5,\ N_{SE} = 94$

Results (2D)

$\Omega = \{x \in \mathbf{R}^2 \, : \, \|x\| \leq 1\}$

$\Omega_p =$ discretization of Ω with 8 elements of order p

$E_S = |\int_{\partial\Omega_p} dx - \int_{\partial\Omega} dx|$

$E_A = |\int_{\Omega_p} dx - \int_{\Omega} dx|$

p	E_S	E_A
1	2e-1	3e-1
2	4e-3	5e-3
3	2e-5	7e-5
4	3e-7	9e-7
5	6e-9	1e-8
6	7e-11	1e-10
7	1e-12	2e-12
8	4e-15	4e-14

Results (3D)

$\Omega = \{x \in \mathbf{R}^3 \,:\, \|x\| \leq 1\}$

$\Omega_p =$ discretization of Ω with 20 (,80) surface elements

$E_S = |\int_{\partial\Omega_p} dx - \int_{\partial\Omega} dx|$

$E_V = |\int_{\Omega_p} dx - \int_{\Omega} dx|$

p	E_S^{20}	E_V^{20}	E_S^{80}	E_V^{80}
1	3	2	9e-1	5e-1
2	4e-1	2e-1	3e-2	2e-2
3	2e-2	1e-2	3e-4	3e-4
4	1e-3	1e-3	7e-6	3e-6
5	8e-5	7e-5	7e-8	4e-8
6	2e-5	1e-5	1e-7	7e-8
7	1e-7	2e-7	8e-10	4e-10
8	3e-7	1e-7	6e-10	3e-10
9	2e-8	1e-8	2e-11	9e-12
10	6e-9	3e-9	4e-12	2e-12
15	7e-13	5e-13	1e-13	5e-14
20	2e-13	4e-14	1e-13	5e-14

Results

Laplace problem:

$$
\begin{aligned}
-\Delta u &= 6 \quad \text{in } \Omega = \{x \in R^3 \, : \, \|x\| \leq 1\} \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Exact solution:

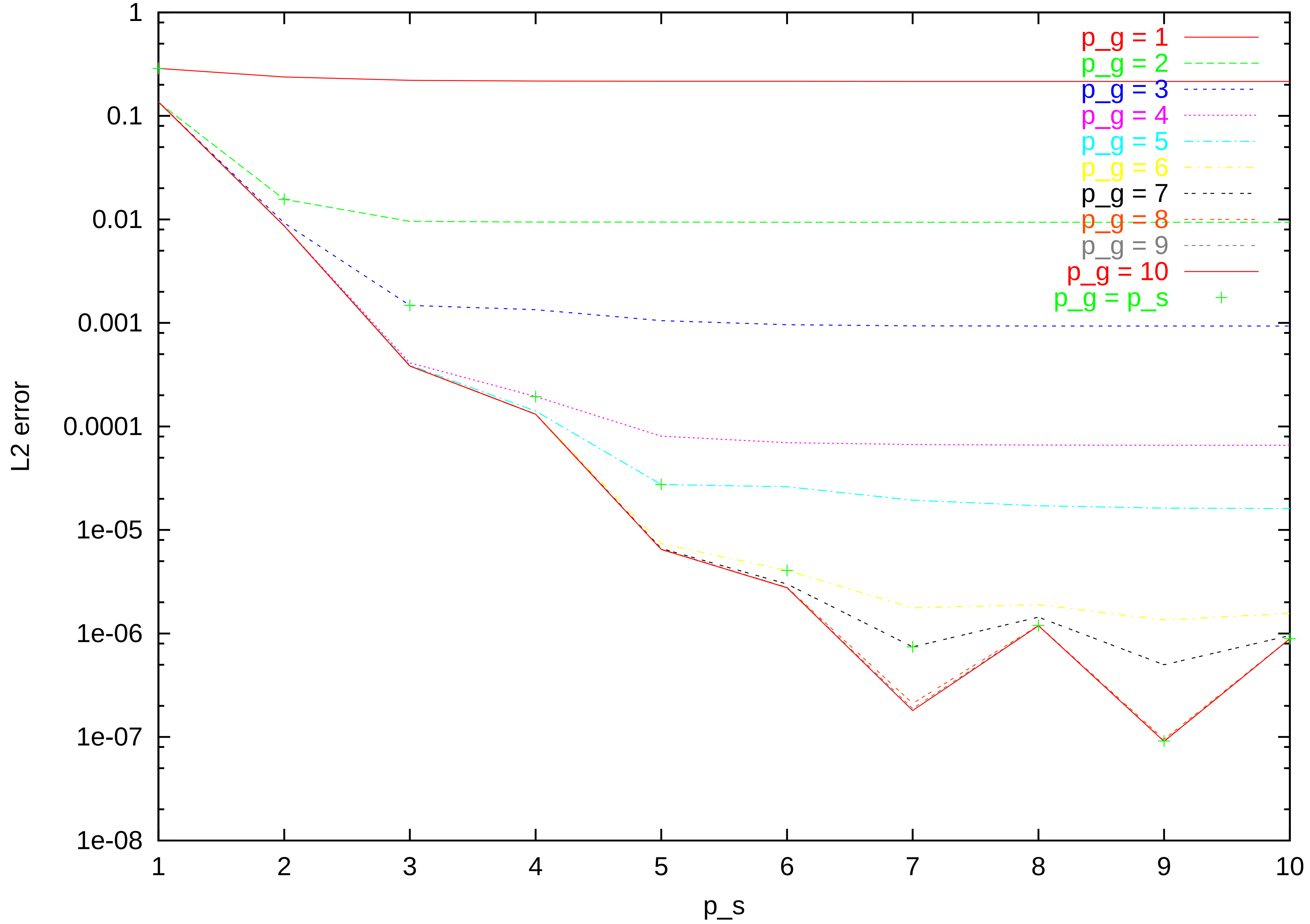
$$u = x^2 + y^2 + z^2 - 1$$

Finite element discretization:

64 curved elements (polynomial degree $p_g = 1, ..., 10$)
high order elements (polynomial degree $p_s = 1, ..., 10$)
gives approximate solution u_{p_g,p_s}

Error:

$$E_{p_g,p_s} = \|u_{p_g,p_s} - u\|_{L^2(\Omega)}$$


p_g = 1
p_g = 2
p_g = 3
p_g = 4
p_g = 5
p_g = 6
p_g = 7
p_g = 8
p_g = 9
p_g = 10
p_g = p_s
L2 error
p_s
1
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
1
2
3
4
5
6
7
8
9
10

Open work

Adaption to square, prism, pyramid