Algebraic Multigrid Methods for Maxwell Equations

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2. Discrete Differential Operators

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Equations of Magnetostatics

Given:

\[ j \text{ .. current density s.t. } \text{div} \, j = 0 \]

Compute:

\[ B \text{ .. magnetic flux density} \]
\[ H \text{ .. magnetic field intensity} \]

such that

\[ B = \mu H \quad \text{div} \, B = 0 \quad \text{curl} \, H = j \]

with the boundary conditions

either \[ B \cdot n = 0 \quad \text{or} \quad H \times n = 0 \]
Coil on a high permeable core
**Vector potential formulation**

Since $\text{div } B = 0$ (plus compatibility conditions), there exists a vector potential $A$ such that

$$B = \text{curl } A$$

Combining the equations above gives us

$$\text{curl } \mu^{-1} \text{curl } A = j$$

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Since $\text{div} \, B = 0$ (plus compatibility conditions), there exists a vector potential $A$ such that

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The weak formulation is to find $A \in V := H(\text{curl})$ such that

$$\int \mu^{-1} \text{curl} \, A \, \text{curl} \, v \, dx = \int j \, v \, dx \quad \forall \, v \in V.$$
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Problem: $A$ is defined up to a $\nabla \varphi$!
Gauging possibilities

1. Do not gauge, work on factor space $H(\text{curl})/\nabla H^1$.
   
   Fine, if error estimates etc. only depend on $\text{curl}A$
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2. Gauging by regularization. Add small $L_2$-term:

   $$
   \int \mu^{-1} \text{curl} A \text{curl} v \, dx + \varepsilon \int A v \, dx = \int j v \, dx \quad \forall v \in V.
   $$

   Fine, if error estimates etc. do not depend on $\varepsilon$.  

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   \]
   Fine, if error estimates etc. do not depend on $\varepsilon$.

3. Gauging by explicit constraints, i.e., solve the mixed problem:
   \[
   \int \mu^{-1} \text{curl} A \text{curl} v \, dx + \int v \nabla \phi \, dx = \int j v \, dx \quad \forall v \in H(\text{curl})
   \]
   \[
   \int A \nabla \psi \, dx = 0 \quad \forall \psi \in H^1
   \]
   Fine, if you like saddle point problems.
The Challenge

Maxwell problems are typically

1. Real three dimensional problems
2. Very ill conditioned (two scales)
3. Have large jumps in the coefficients (permeability $\mu_{rel} = 1 \ldots 10^4$, conductivity $\sigma$)
4. Have complicated and nasty geometry (thin shields)
5. Show thin boundary layers (Eddy current problems)
6. Are indefinite for high frequencies (not discussed here)
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The equation solver is a big challenge.

Black box iterative solvers are a mess (SSOR, Incomplete Cholesky, standard AMG, ...)

Joachim Schöberl
Maxwell equations and finite elements
Function Spaces

\[ L_2 := \{ v : \int v^2 \, dx < \infty \} \]
\[ H^1 := \{ v \in L_2 : \text{grad} \, v \in [L_2]^3 \} \]
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\[ H^1 := \{ v \in L_2 : \text{grad} \, v \in [L_2]^3 \} \]

\[ H(\text{curl}) := \{ v \in [L_2]^3 : \text{curl} \, v \in [L_2]^3 \} \]

\[ H(\text{div}) := \{ v \in [L_2]^3 : \text{div} \, v \in L_2 \} \]
Function Spaces

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\[ H^1 := \{ v \in L_2 : \text{grad} \, v \in [L_2]^3 \} \]
\[ H(\text{curl}) := \{ v \in [L_2]^3 : \text{curl} \, v \in [L_2]^3 \} \]
\[ H(\text{div}) := \{ v \in [L_2]^3 : \text{div} \, v \in L_2 \} \]

These spaces form a complete sequence:

\[ H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L_2 \]

There is

\[ \text{grad} \, H^1 = \{ v \in H(\text{curl}) : \text{curl} \, v = 0 \} \]
\[ \text{curl} \, H(\text{curl}) = \{ v \in H(\text{div}) : \text{div} \, v = 0 \} \]
The mesh topology

Tetrahedral mesh with

set of vertices \( \mathcal{V} = \{ V_i \} \),
set of edges \( \mathcal{E} = \{ E_{ij} \} \),
set of faces \( \mathcal{F} = \{ F_{ijk} \} \),
set of cells \( \mathcal{C} = \{ C_{ijkl} \} \).

These entities are used to define the finite element degrees of freedom (= evaluation functionals):

- **Vertex values**: \( v(V_i) \)
- **Edge integrals**: \( \int_{E_{ij}} \tau \cdot v \, ds \)
- **Face integrals**: \( \int_{F_{ijk}} \nu \cdot v \, ds \)
- **Cell integrals**: \( \int_{C_{ijkl}} v \, dx \)
The de Rham complex

\[ H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \]

\[ V^v \xrightarrow{\text{grad}} V^e \xrightarrow{\text{curl}} V^f \xrightarrow{\text{div}} V^c \]

Nédélec \quad Raviart-Thomas
The de Rham complex

$$
\begin{array}{cccc}
H^1 & \xrightarrow{\text{grad}} & H(\text{curl}) & \xrightarrow{\text{curl}} & H(\text{div}) & \xrightarrow{\text{div}} & L^2 \\
\cup & \cup & \cup & \cup & \cup & \cup & \\
V^v & \xrightarrow{\text{grad}} & V^e & \xrightarrow{\text{curl}} & V^f & \xrightarrow{\text{div}} & V^c
\end{array}
$$

- basic properties: Bossavit, Hiptmair
- a-priori estimates: Monk, Vardapetyan-Demkowicz, Nicaise, Schöberl
- Eigenvalue problems: Kikuchi, Boffi, Demkowicz-Monk-Schwab-Vardapetyan
- Multigrid and domain decomposition: Arnold-Falk-Winther, Hiptmair, Toselli
- A posteriori error estimates: Beck-Hiptmair-Hoppe-Wohlmuth
The de Rham complex

\[
\begin{array}{c}
H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \\
\downarrow \Pi^v \quad \downarrow \Pi^e \quad \downarrow \Pi^f \quad \downarrow \Pi^c \\
V^v \xrightarrow{\text{grad}} V^e \xrightarrow{\text{curl}} V^f \xrightarrow{\text{div}} V^c
\end{array}
\]

Nédélec \quad Raviart-Thomas

The interpolation operators fulfill the commuting diagram properties

\[
\text{grad} \Pi^v = \Pi^e \text{grad} \quad \text{curl} \Pi^e = \Pi^f \text{curl} \quad \text{div} \Pi^f = \Pi^c \text{div}
\]
Discrete Gradient Operator

Take $w \in V^v \subset H^1$. Its expansion in the canonical basis (hat functions) is

$$w(x) = \sum_{V_i \in \mathcal{V}} w_i \varphi_i^v(x) \quad \text{with} \quad w_i = w(V_i)$$

The same for $v \in V^e \subset H(\text{curl})$:

$$v(x) = \sum_{E_{ij} \in \mathcal{E}} v_{ij} \varphi_{ij}^e(x) \quad \text{with} \quad v_{ij} = \int_{V_i}^{V_j} \tau \cdot v \, ds$$
Discrete Gradient Operator

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$$w(x) = \sum_{V_i \in V} w_i \varphi_i^v(x) \quad \text{with} \quad w_i = w(V_i)$$

The same for $v \in V^e \subset H(\text{curl})$:

$$v(x) = \sum_{E_{ij} \in E} v_{ij} \varphi_{ij}^e(x) \quad \text{with} \quad v_{ij} = \int_{V_i}^{V_j} \tau \cdot v \, ds$$

Now, for $v = \nabla w$ there is

$$v_{ij} = \int_{V_i}^{V_j} \tau \cdot \nabla w \, ds = w(V_j) - w(V_i) = w_j - w_i$$
Discrete Gradient Operator

Take \( w \in V^v \subset H^1 \). Its expansion in the canonical basis (hat functions) is

\[
w(x) = \sum_{V_i \in V} w_i \phi^v_i(x) \quad \text{with} \quad w_i = w(V_i)
\]

The same for \( v \in V^e \subset H(\text{curl}) \):

\[
v(x) = \sum_{E_{ij} \in \mathcal{E}} v_{ij} \phi^e_{ij}(x) \quad \text{with} \quad v_{ij} = \int_{V_i}^{V_j} \tau \cdot v \, ds
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Now, for \( v = \nabla w \) there is

\[
v_{ij} = \int_{V_i}^{V_j} \tau \cdot \nabla w \, ds = w(V_j) - w(V_i) = w_j - w_i
\]

With a matrix \( B_{\text{grad}} \in \mathbb{R}^{N_e \times N_v} \) we write \( v = B_{\text{grad}} w \).

\[
[B_{\text{grad}}]_{E_{ij},V_k} = \begin{cases} 
1 & \text{for } k = j \\
-1 & \text{for } k = i \\
0 & \text{else}
\end{cases}
\]
We continue with $q \in V^f \subset H(\text{div})$:

$$q(x) = \sum_{F_{ijk} \in \mathcal{F}} q_{ijk} \varphi_{ijk}^f(x) \quad \text{with} \quad q_{ijk} = \int_{F_{ijk}} \nu \cdot q \, ds$$
Discrete Curl Operator

We continue with $q \in V^f \subset H(\text{div})$:

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For $q = \text{curl} \, v$, Stokes’ Theorem gives:

$$q_{ijk} = \int_{F_{ijk}} \nu \cdot \text{curl} \, v \, ds = \int_{\partial F_{ijk}} \tau \cdot v \, ds$$

$$= \int_{E_{ij}} + \int_{E_{jk}} + \int_{E_{ki}} \tau \cdot v \, ds = v_{ij} + v_{jk} + v_{ki}$$
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$$= \int_{E_{ij}} + \int_{E_{jk}} + \int_{E_{ki}} \tau \cdot \nu \, ds = v_{ij} + v_{jk} + v_{ki}$$

With a matrix $B_{\text{curl}} \in \mathbb{R}^{N_f \times N_e}$ we write $q = B_{\text{curl}} \nu$.

$$[B_{\text{curl}}]_{F_{ijk}, E_{lm}} = \begin{cases} 
1 & \text{for } ij = lm \text{ or } jk = lm \text{ or } ki = lm \\
-1 & \text{for } ij = ml \text{ or } jk = ml \text{ or } ki = ml \\
0 & \text{else} \end{cases}$$
Discrete Div Operator

And, finally $s \in V^c \subset L_2$:

$$s(x) = \sum_{C_{ijkl} \in \mathcal{C}} s_{ijkl} \varphi_{ijkl}^c(x) \quad \text{with} \quad s_{ijkl} = \int_{C_{ijkl}} s \, dx$$
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For $s = \text{div} \, q$, Gauss' Theorem gives:

$$s_{ijkl} = \int_{C_{ijkl}} \text{div} \, q \, ds = \int_{\partial C_{ijkl}} \nu \cdot q \, ds$$

$$= \int_{F_{ijk}} + \int_{F_{lij}} + \int_{F_{kli}} + \int_{F_{jkl}} \nu \cdot q \, ds = q_{ijk} + v_{lij} + v_{kli} + v_{jkl}$$
Discrete Div Operator

And, finally $s \in V^e \subset L_2$:

$$s(x) = \sum_{C_{ijkl} \in \mathcal{C}} s_{ijkl} \varphi_{ijkl}^c(x) \quad \text{with} \quad s_{ijkl} = \int_{C_{ijkl}} s \, dx$$

For $s = \text{div} \, q$, Gauss’ Theorem gives:

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With a matrix $B_{\text{div}} \in \mathbb{R}^{N_e \times N_f}$ we write $s = B_{\text{div}}q$. 
Mass matrices

For all spaces $V^v$, $V^e$, $V^f$, $V^c$, we define the Gramian matrices (mass matrices)

\[
[M^v_\lambda]_{V, V'} = \int \lambda(x) \varphi^v_V(x) \varphi^v_{V'}(x) \, dx
\]

\[
[M^e_\lambda]_{E, E'} = \int \lambda(x) \varphi^e_E(x) \varphi^e_{E'}(x) \, dx
\]

\[
[M^f_\lambda]_{F, F'} = \int \lambda(x) \varphi^f_F(x) \varphi^f_{F'}(x) \, dx
\]

\[
[M^c_\lambda]_{C, C'} = \int \lambda(x) \varphi^c_C(x) \varphi^c_{C'}(x) \, dx
\]

For all these matrices, diagonal preconditioning is optimal:

\[
\text{cond}((\text{diag}[M])^{-1}M) \simeq 1
\]

Some of the constants depend on the maximal angle of the elements.
System matrices

Indeed, we want to discretise variational forms such as

\[ \int \nu \text{curl } u \text{ curl } v \, dx + \int \sigma uv \, dx \]

With the discrete differential operator \( B_{\text{curl}} \) and the mass matrices, the system matrix is

\[ A^e = B_{\text{curl}}^T M^f_{\nu} B_{\text{curl}} + M^e_{\sigma} \]
System matrices

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\[ \int \nu \text{curl} \ u \ \text{curl} \ v \ dx + \int \sigma uv \ dx \]

With the discrete differential operator \( B_{\text{curl}} \) and the mass matrices, the system matrix is

\[ A^e = B_{\text{curl}}^T M_\nu B_{\text{curl}} + M_\sigma \]

Remark: For all matrices there is a lumped approximation

\[ A^v \simeq B_{\text{grad}}^T \text{diag}[M^e] B_{\text{grad}} + \text{diag}[M^v] \]

This approximation is an M-matrix:

\[ u^T A^v u = \sum_{E_{ij}} M_{ij}^e (u_i - u_j)^2 + \sum_{V_i} M_i^v u_i^2 \]

Can be used to control coarsening
Smoothing iterations

Consider the $H^1$ elliptic form
\[ \int \nabla u \nabla v \, dx + \varepsilon \int uv \, dx, \]
leading to the matrix
\[ A^v = B^T_{\text{grad}} M^e_{1} B_{\text{grad}} + M^v_{\varepsilon} \]
This matrix has one small eigenvalue, the corresponding eigenvector is the constant.

Diagonal preconditioning with $C = \text{diag} A^v$ is not robust
\[ \text{cond}\{C^{-1} A^v\} \simeq \varepsilon^{-1} h^{-2} \]
The problem is the kernel space. Define one more \( B \):

\[
B_{id} \in \mathbb{R}^{N_v \times 1} \quad [B_{id}]_{V,1} = 1
\]

Thus

\[
\text{kernel}(B_{\text{grad}}) = \text{range}(B_{id})
\]

Knowing the kernel explicitly, we can improve the preconditioner on the kernel:

\[
C^{-1} = \text{diag}(A^v)^{-1} + B_{id}(\text{diag}(B_{id}^T A^v B_{id}))^{-1} B_{id}^T
\]

The new preconditioner is robust in \( \varepsilon \):

\[
\text{cond}\{ C^{-1} A^v \} \sim h^{-2}
\]
Hiptmair’s smoother

The problem is (in principle) the same for $H(\text{curl})$:

$$A^e = B^T_{\text{curl}}M_{\nu}B_{\text{curl}} + M^e_{\sigma}$$

The leading term has a \textit{large} kernel. Thanks to the complete sequence property it is known explicitly:

$$\text{kernel}(B_{\text{curl}}) = \text{range}(B_{\text{grad}})$$

Thus, Hiptmair added additional smoothing steps in the kernel:

$$C^{-1} = \text{diag}(A^e)^{-1} + B_{\text{grad}}(\text{diag}(B^T_{\text{grad}}A^eB_{\text{grad}}))^{-1}B^T_{\text{grad}}$$
Hiptmair’s smoother

The problem is (in principle) the same for $H(\text{curl})$:

$$A^e = B_{\text{curl}}^T M^f B_{\text{curl}} + M^e_\sigma$$

The leading term has a large kernel. Thanks to the complete sequence property it is known explicitely:

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$$C^{-1} = \text{diag}(A^e)^{-1} + B_{\text{grad}} \left( \text{diag}(B_{\text{grad}}^T A^e B_{\text{grad}}) \right)^{-1} B_{\text{grad}}^T$$

The additional problem is of Poisson type:

$$B_{\text{grad}}^T A^e B_{\text{grad}} = B_{\text{grad}}^T B_{\text{curl}}^T M^f B_{\text{curl}} B_{\text{grad}} + B_{\text{grad}}^T M^e_\sigma B_{\text{grad}} = 0$$
Algebraic coarsening based on Agglomeration

Defined by the mapping 
\[ \text{Ind}(.) : \text{Vertex} \rightarrow \text{Cluster} \]

Allows to define the full coarse grid topology:

- \( E_{IJ} \) is a coarse grid edge if and only if there are fine grid vertices \( i \) and \( j \) s.t.:
  \[ I = \text{Ind}(i), \quad J = \text{Ind}(j), \quad E_{ij} \text{ is a fine grid edge} \]

- \( F_{IJK} \) is a coarse grid face if and only if there are fine grid vertices \( i, j, \) and \( k \) s.t.:
  \[ I = \text{Ind}(i), \quad J = \text{Ind}(j), \quad K = \text{Ind}(k), \quad F_{ijk} \text{ is a fine grid face} \]
Coarse grid spaces

Vertex coarse grid space (constant in cluster):

\[ V_{\text{coarse}}^v = \{ v \in V^v : v(V^i) = v(V^{i'}) \text{ for } \text{Ind}(i) = \text{Ind}(i') \}, \]

the prolongation matrix \( P^v \in \mathbb{R}^{N_v \times N_{v,\text{coarse}}} \) is

\[
[P^v]_{i,I} = \begin{cases} 
1 & \text{if } I = \text{Ind}(i) \\
0 & \text{else}
\end{cases}
\]
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Vertex coarse grid space (constant in cluster):

\[ V^v_{\text{coarse}} = \{ v \in V^v : v(V^i) = v(V^{i'}) \text{ for } \text{Ind}(i) = \text{Ind}(i') \}, \]

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\[
[P^v]_{i,I} = \begin{cases} 
1 & \text{if } I = \text{Ind}(i) \\
0 & \text{else}
\end{cases}
\]

Edge coarse grid space:

\[ V^e_{\text{coarse}} = \{ v \in V^e : \int_{E_{ij}} \tau v \, ds = \int_{E_{i'j'}} \tau v \, ds \text{ for } \text{Ind}(i) = \text{Ind}(i'), \text{Ind}(j) = \text{Ind}(j') \}, \]

the prolongation matrix \( P^e \in \mathbb{R}^{N_e \times N_e,\text{coarse}} \) is

\[
[P^e]_{ij,IJ} = \begin{cases} 
1 & \text{if } I = \text{Ind}(i) \text{ and } J = \text{Ind}(j) \\
-1 & \text{if } I = \text{Ind}(j) \text{ and } J = \text{Ind}(i) \\
0 & \text{else}
\end{cases}
\]
Coarse grid differential operators

Gradients of coarse grid functions in $V^v_{coarse} \subset V^v$:

Coarse vertex basis function

Its gradient in $V^e$

Coarse edge basis function
Gradients of coarse grid functions in $V_{coarse}^v \subset V^v$:

Coarse vertex basis function

Its gradient in $V^e$

Coarse edge basis function

There holds $\text{grad} V_{coarse}^v \subset V_{coarse}^e$, i.e. for any $w_{coarse}^v$, there exists an unique $v_{coarse}^e$ such that

$$B_{\text{grad}} P^v w_{coarse}^v = P^e v_{coarse}^e.$$  

This allows the definition of $B_{\text{grad}}^{coarse}$. 

Coarse grid differential operators
The 2-Level de Rham diagram:

$$
\begin{align*}
V^v & \xrightarrow{B_{\text{grad}}} V^e & B_{\text{curl}} & V^f & B_{\text{div}} & V^c \\
\downarrow \Pi^v & \downarrow \Pi^e & \downarrow \Pi^f & \downarrow \Pi^c
\end{align*}
$$

$$
\begin{align*}
V_{\text{coarse}}^v & \xrightarrow{B_{\text{grad}}} V_{\text{coarse}}^e & B_{\text{curl}} & V_{\text{coarse}}^f & B_{\text{div}} & V_{\text{coarse}}^c
\end{align*}
$$

1. The algebraically constructed coarse spaces form a complete sequence.

2. There are commuting interpolation operators

3. The interpolation operators are $L_2$ bounded

4. Statement 2 and 3 imply that $\Pi^v$, $\Pi^e$, and $\Pi^f$ are bounded in $H^1$, $H(\text{curl})$, and $H(\text{div})$ -norms, respectively. This is essential for the two-level analysis [Reitzinger-Sch., 2001].
Model Problem

\[ \Omega = (0,1)^3, \quad V = H_0(\text{curl}), \quad f = (1,0,0), \]

Variational form:

\[ \int \text{curl} \, u, \text{curl} \, v \, dx + 10^{-3} \int uv \, dx = \int fv \, dx \]

V-11 cycle:

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<th>( N_h^c )</th>
<th>setup (sec)</th>
<th>solver (sec)</th>
<th>iteration</th>
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Variable V cycle:

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</table>

Computation with Stefan Reitzinger’s AMG code Pebbles, CPU = PIII 1 GHz
TEAM 20 Benchmark problem

Coil and Iron core, small air gap.

Unknovns: 240E3
Iterations: 26
Solution time: 90 sec

by Manfred Kaltenbacher,
University Erlangen, Germany
Using the code Pebbles
Simulation of a Transformer

Project with EBG Transformatorenbau
Three phase transformer

- prescribed current sources in coils
- main flux though high permeable core
- flux penetrating the casing causes eddy currents
- thin shields collecting stray fluxes

Model:
- Time harmonic, low frequency
- Nonlinear terms due to saturation in casing
Magnetic flux density
Loss density in pressing plates
Eddy current density
Some remarks

• The problems have 2 coefficients. Which one should drive the coarsening? Recursive AMG!

• Clustering is based on diagonal entries of the mass matrices of derivatives.

• Are better coarsening strategies possible?