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Multigrid Solvers for Multiharmonic Nonlinear Magnetic Field Computations

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Problem Formulation

Maxwell's equation in the quasi-stationary case, i.e. for eddy current problems, read as follows:

$$\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J}\,,\qquad \mathbf{curl}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\,,\qquad \operatorname{div}\boldsymbol{B} = 0\,,$$

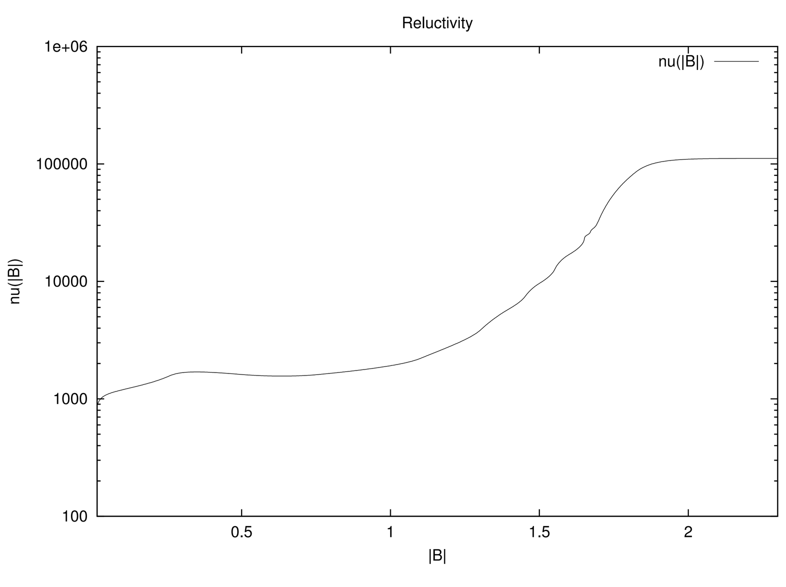
with the electric field $\boldsymbol{E}$, the magnetic field $\boldsymbol{H}$, the induction $\boldsymbol{B}$ and the electric current density $\boldsymbol{J}$.
Together with the relation $\boldsymbol{H} = \nu(|\boldsymbol{B}|)\boldsymbol{B}$, the equations transform to

$$\sigma\frac{\partial \boldsymbol{u}}{\partial t} + \mathbf{curl}\,(\nu(|\mathbf{curl}\,\boldsymbol{u}|)\,\mathbf{curl}\,\boldsymbol{u}) = \boldsymbol{f},$$

where $\boldsymbol{u}$ is the vector potential and $\boldsymbol{f}$ denotes the sources. We regard this equation in $\Omega\times[0,T]$ and add the boundary condition $\boldsymbol{u}\times\boldsymbol{n} = 0$ on $\Gamma\times[0,T]$ and the initial condition $\boldsymbol{u} = \boldsymbol{u}_0$ on $\Omega\times\{0\}$.

B-*H*-Curve

The relation between $\boldsymbol{B}$ and $\boldsymbol{H}$ is given by a set of measured data points, which are approximated by splines.
Due to the physical background the function $s \mapsto \nu(s)\cdot s$ is strictly monotone, and in our approximation we have $\nu \in C^1(\mathbb{R}_0^+)$.



Reluctivity $\nu(|\boldsymbol{B}|)$ for a ferromagnetic material

Existence and Uniqueness

Eddy current problems are essentially different for conducting ($\sigma > 0$) and non-conducting regions ($\sigma = 0$).

In **non-conducting regions**, we face the equation

$$\int_\Omega \nu\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} = \int_\Omega \boldsymbol{f}\,\boldsymbol{v}, \qquad \forall\boldsymbol{v}\in\boldsymbol{V},$$

with $\boldsymbol{V} = \boldsymbol{H}_0(\mathbf{curl})$.
This problem is uniquely solvable in the space of divergence-free functions, i.e. in $\bar{\boldsymbol{V}} := \boldsymbol{V}/\boldsymbol{W}$ with

$$\boldsymbol{W} := \{\boldsymbol{w} = \mathbf{grad}\,\phi : \phi\in H^1(\Omega) \text{ and } \phi = c_i \text{ on } \Gamma_i,\ 1\le i\le p\}.$$

Here, for multiply connected domains, we denote the $p \ge 1$ components of the boundary $\partial\Omega$ by Γ_i, $1 \le i \le p$.

In **conducting regions**, we are confronted with the parabolic problem

Find $\boldsymbol{u}\in L_2((0,T),\bar{\boldsymbol{V}})$ with $\dot{\boldsymbol{u}}\in L_2((0,T),\bar{\boldsymbol{V}}^*)$:

$$\sigma\,\dot{\boldsymbol{u}} + A(\boldsymbol{u}) = F,\qquad \boldsymbol{u}(0) = \boldsymbol{u}_0,$$

with the operators

$$\langle A(\boldsymbol{u})(t),\boldsymbol{v}\rangle := \int_\Omega \nu(|\mathbf{curl}\,\boldsymbol{u}(t)|)\,\mathbf{curl}\,\boldsymbol{u}(t)\cdot\mathbf{curl}\,\boldsymbol{v},$$
$$\langle F(t),\boldsymbol{v}\rangle := \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}.$$

This problem is uniquely solvable for $s\mapsto\nu(s)s$ monotone, where $\bar{\boldsymbol{V}} = \boldsymbol{V}/\boldsymbol{W}$ as above.

For divergence-free current source $\boldsymbol{f}$ and initial condition $\boldsymbol{u}_0$, the solution is even unique in the space $W_2^1((0,T),\boldsymbol{H}_0(\mathbf{curl}))$.

For **general eddy current problems**, unique solvability follows by connecting these two results.

Multiharmonic Ansatz

Since we have, $\boldsymbol{f} = \boldsymbol{f}^c\cdot\cos(\omega t)$, the ansatz $\boldsymbol{u} = \boldsymbol{u}^c\cdot\cos(\omega t) + \boldsymbol{u}^s\cdot\sin(\omega t)$ seems to be obvious. Because of the nonlinear reluctivity ν, however, the solution is not necessarily harmonic, but still periodical. This gives rise to the idea of a multiharmonic ansatz

$$\boldsymbol{u}(\boldsymbol{x},t) \sim \sum_{k=0}^{N}\left[\boldsymbol{u}_k^c(\boldsymbol{x})\cdot\cos(k\omega t) + \boldsymbol{u}_k^s(\boldsymbol{x})\cdot\sin(k\omega t)\right].$$

The same ansatz is used for the magnetic field $\boldsymbol{H}$.
It can be shown that all even Fourier coefficients of $\boldsymbol{u}$ and $\boldsymbol{H}$ vanish, provided the same holds for $\boldsymbol{f}$.

Applying this ansatz to the original equation and testing with $\cos(m\omega t)$ and $\sin(m\omega t)$ leads to

$$\mathbf{curl}\begin{pmatrix}\boldsymbol{H}_1^c(\mathbf{curl}\,\boldsymbol{u})\\ \boldsymbol{H}_1^s(\mathbf{curl}\,\boldsymbol{u})\\ \vdots\\ \boldsymbol{H}_{2n+1}^c(\mathbf{curl}\,\boldsymbol{u})\\ \boldsymbol{H}_{2n+1}^s(\mathbf{curl}\,\boldsymbol{u})\end{pmatrix} + \omega\sigma\underbrace{\begin{pmatrix}0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 2n+1\\ & & & -(2n+1) & 0\end{pmatrix}}_{=:\boldsymbol{D}}\begin{pmatrix}\boldsymbol{u}_1^c\\ \boldsymbol{u}_1^s\\ \vdots\\ \boldsymbol{u}_{2n+1}^c\\ \boldsymbol{u}_{2n+1}^s\end{pmatrix} = \begin{pmatrix}\boldsymbol{f}_1^c\\ \boldsymbol{f}_1^s\\ \vdots\\ \boldsymbol{f}_{2n+1}^c\\ \boldsymbol{f}_{2n+1}^s\end{pmatrix},$$

where we assume $N = 2n+1$.
With the abbreviations $\boldsymbol{H} = (\boldsymbol{H}_1^c,\ \boldsymbol{H}_1^s,\ \ldots)^T$, and analoguously $\boldsymbol{u}$ and $\boldsymbol{f}$, the problem can be written in a more compact way:

$$\mathbf{curl}\,\boldsymbol{H}(\mathbf{curl}\,\boldsymbol{u}) + \omega\sigma\,\boldsymbol{D}\,\boldsymbol{u} = \boldsymbol{f},$$

or in weak formulation

$$\underbrace{\int_\Omega \mathbf{curl}\,\boldsymbol{v}^T\boldsymbol{H}(\mathbf{curl}\,\boldsymbol{u}) + \omega\sigma\,\boldsymbol{v}^T\boldsymbol{D}\,\boldsymbol{u}}_{=:\langle A(\boldsymbol{u}),\boldsymbol{v}\rangle} = \underbrace{\int_\Omega \boldsymbol{v}^T\boldsymbol{f}}_{=:\langle F,\boldsymbol{v}\rangle},\quad \forall\boldsymbol{v}.$$

Regularization

For the numerical solution of the variational problem, a regularization parameter ϵ is introduced. More precisely, we set $\sigma = \epsilon > 0$ in the non-conducting regions. This perturbed problem is uniquely solvable in the space $\boldsymbol{H}_0(\mathbf{curl})^{N+1}$.

The procedure is justified, because the equation is equivalent to a uniquely solvable mixed problem. The latter can be perturbed without changing its structure.

Consider, for example, the linear harmonic problem

$$a(\boldsymbol{u},\boldsymbol{v}) := \langle A\boldsymbol{u},\boldsymbol{v}\rangle = \langle F,\boldsymbol{v}\rangle,\quad \forall\boldsymbol{v}\in\boldsymbol{V}. \qquad (1)$$

Denote the gradient fields by $\boldsymbol{W}$. Then (1) is equivalent to the mixed problem

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},\boldsymbol{\phi}) &= \langle F,\boldsymbol{v}\rangle, && \forall\boldsymbol{v}\in\boldsymbol{V},\\ b(\boldsymbol{u},\boldsymbol{\psi}) &= 0, && \forall\boldsymbol{\psi}\in\boldsymbol{W},\end{aligned}\qquad (2)$$

with the bilinear form $b(\boldsymbol{v},\boldsymbol{\phi}) = \int \boldsymbol{v}^T\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}$, since we have

$$\boldsymbol{u} \text{ solves } (1) \implies (\boldsymbol{u},0) \text{ solves } (2),$$
$$(\boldsymbol{u},\boldsymbol{\phi}) \text{ solves } (2) \implies \boldsymbol{u} \text{ solves } (1).$$

With the perturbed bilinear form $a_\epsilon(\cdot,\cdot)$, we have the following estimate (Strang):

$$\|\boldsymbol{u}-\boldsymbol{u}_\epsilon\|_V \le C\inf_{\boldsymbol{v}\in\boldsymbol{V}/\boldsymbol{W}}\left[\|\boldsymbol{u}-\boldsymbol{v}\|_V + \sup_{\boldsymbol{w}\in\boldsymbol{V}/\boldsymbol{W}}\frac{a(\boldsymbol{v},\boldsymbol{w}) - a_\epsilon(\boldsymbol{v},\boldsymbol{w})}{\|\boldsymbol{w}\|_V}\right],$$

where $\boldsymbol{u}$ is the solution of (2) $\Leftrightarrow$ (1) and $\boldsymbol{u}_\epsilon$ the solution of the perturbed problem.
If for the perturbed bilinear form, we replace $\sigma = 0$ in $\Omega_2\subset\Omega$ by $\epsilon > 0$, we have

$$a(\boldsymbol{v},\boldsymbol{w}) - a_\epsilon(\boldsymbol{v},\boldsymbol{w}) = -\epsilon\,\omega\int_{\Omega_2}\boldsymbol{w}^T\boldsymbol{D}\boldsymbol{v}.$$

Preconditioning

Finite element discretization of the perturbed linear harmonic problem leads to the system

$$\begin{pmatrix} A & M\\ -M & A\end{pmatrix}\underline{u} = \underline{f}.$$

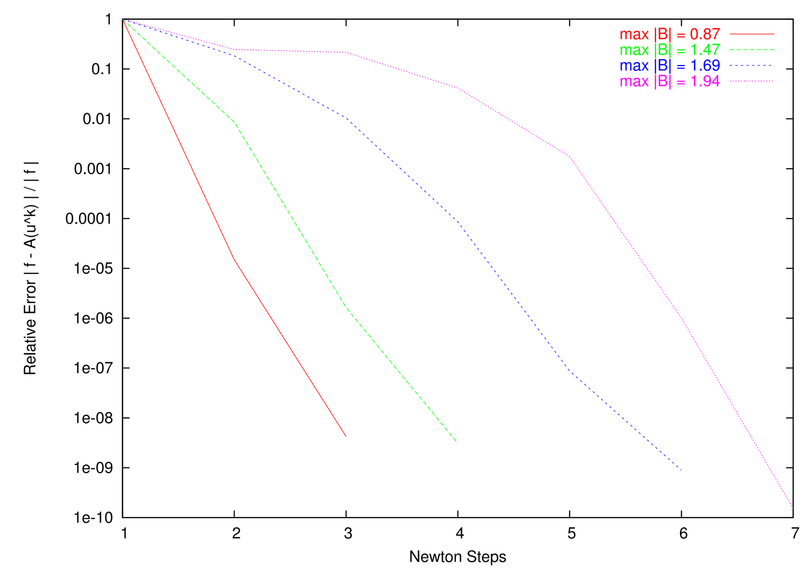
For preconditioning this problem, we use the matrix

$$C^{-1} = \frac{1}{2}\begin{pmatrix}(\widetilde{A+M})^{-1} & \\ & (\widetilde{A+M})^{-1}\end{pmatrix}\begin{pmatrix} I & I\\ I & -I\end{pmatrix},$$

where the approximate inverse of $A + M$ is realized by a multigrid iteration. We solve the linear system by the Quasi-Minimal Residual method.

Newton Iteration

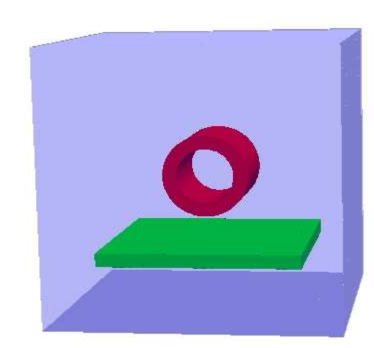
The nonlinear problem is solved iteratively by Newton's method. The convergence speed of this iteration obviously is influenced by the properties of the nonlinear reluctivity ν. Since the relation between $\boldsymbol{B}$ and $\boldsymbol{H}$ is close to linear for small inductions $|\boldsymbol{B}|$, the iteration converges faster in this case. For $\max|\boldsymbol{B}|\sim 2$, the method moderately slows down:

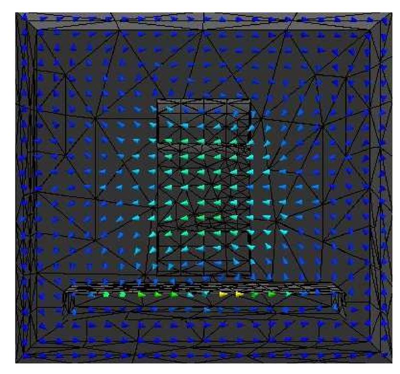


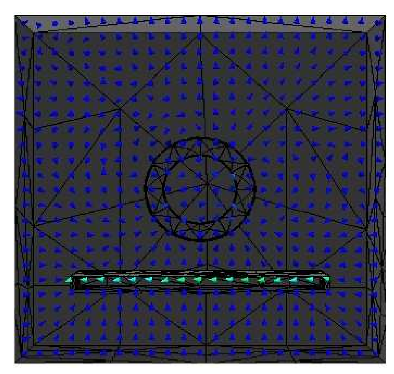
The speed of convergence depends on the maximal induction $|\boldsymbol{B}|$.

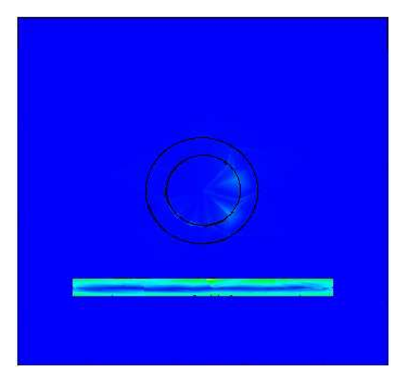
Numerical Results

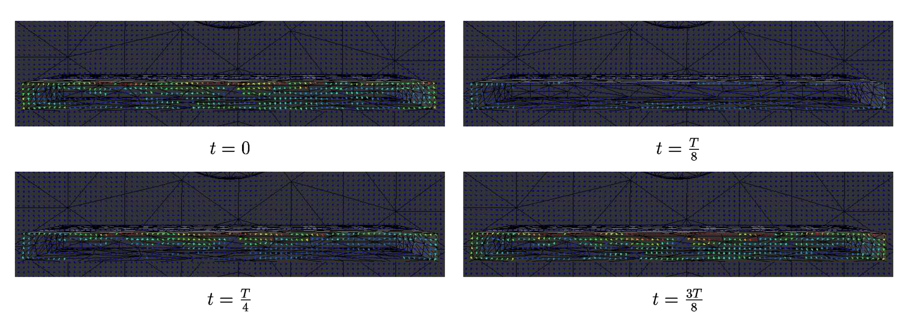
We present some results for a shielding problem and for an eddy current welding problem:


Geometry of a shielding problem.

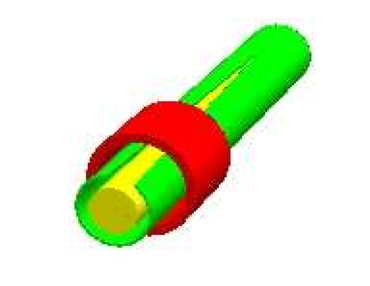

Induction at $t = 0$.

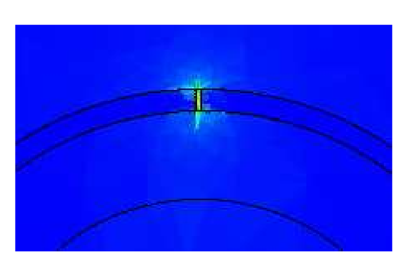

Eddy currents (at $t = 0$).

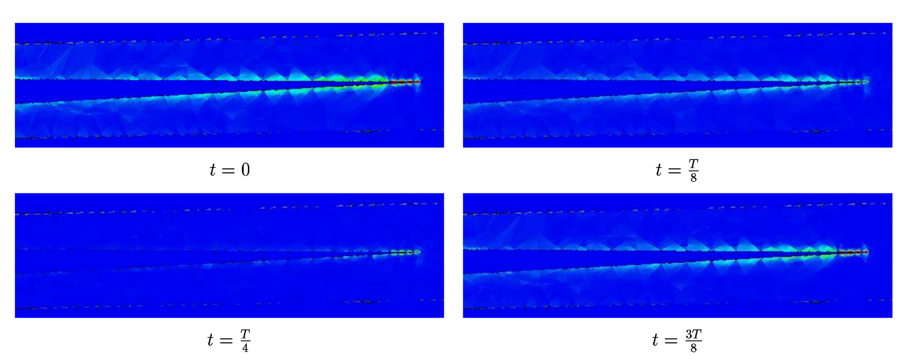

Absolute value of eddy currents (at $t = 0$).


$t = 0$, $t = \frac{T}{8}$, $t = \frac{T}{4}$, $t = \frac{3T}{8}$

Eddy currents for different t (zoomed).


Geometry of an eddy current welding problem.


Absolute value of induction in the slitted tube, zoomed ($t = 0$).


$t = 0$, $t = \frac{T}{8}$, $t = \frac{T}{4}$, $t = \frac{3T}{8}$

Absolute value of eddy currents in the slitted tube, for different t (zoomed).

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