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High order EAS elements for plates and 3D structures

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Abstract

Enhanced Assumed Strain (EAS) elements have been introduced to avoid volume and shear locking problems, and are commonly used nowadays. The EAS technique can be considered as a convenient implementation method for realizing selective projection operators.

In this paper, we present high order basis functions for quadrilateral and triangular EAS elements for the Reissner Mindlin plate model. Additionally, we present a domain decomposition preconditioner for the high order elements, where the condition numbers are uniform in the thickness and grow only slightly with the polynomial order. Finally, we transform the results from the plate model to thin 3D domains.

1 A high order EAS finite element method for the Reissner Mindlin plate

We consider the Reissner Mindlin plate model, which is an equation for the vertical deflection w and the rotations β. We set $W = \{v \in H^1 : v = 0 \text{ on } \Gamma_D\}$, and $V = \{\delta \in H^1 : \delta = 0 \text{ on } \Gamma_D\}^2$, where Γ_D is the part of the boundary where the plate is clamped. The variational formulation is: find $w \in W$ and $\beta \in V$ such that

$$a^b(\beta, \delta) + \frac{E}{t^2} \int_\Omega (\nabla w - \beta) \cdot (\nabla v - \delta)\, dx = \int_\Omega f v \, dx \qquad \forall\, v \in W, \forall\, \delta \in V.$$

Here, $a^b(\beta, \delta) := \int_\Omega D\varepsilon(\beta) : \varepsilon(\delta)\, dx$ is the bending term, t is the thickness, E is the young modulus, and f is the vertical load. A conforming low order finite element method applied to the Reissner Mindlin model suffers from bad accuracy if the thickness t is small. Thus, usually a shear reduction operator is introduced. A high order method converges robust in t, but may loose some order in the convergence. We start with the following method by Chapelle and Stenberg [CS]. The mesh may consist of triangles T and quadrilaterals R, the generic edge is called E. We choose finite element spaces

$$\begin{aligned} W_h &= \{v_h \in W : v_h|_T \in P^{p+1}, v_h|_R \in Q^{p+1}\}, \\ V_h &= \{\delta_h \in V : \delta_h|_T \in P^{p+2}, \delta_h|_R \in P^{p+1}, \delta_h|_E \in P^p\}^2, \end{aligned}$$

where P^p is the space of polynomials up to total order p, and Q^p is the space of polynomials up to order p in each variable. Furthermore, we define the space

$$Q_h = \{q_h \in L_2 : q_h|_T \in P^{p-1}, q_h|_R \in Q^{p-1}\}^2,$$

and the L_2-orthogonal projection $P_{Q_h} : [L_2]^2 \to Q_h : \quad \int_\Omega P_{Q_h}[q] v_h \, dx = \int_\Omega q v_h \, dx \quad \forall v_h \in Q_h$. The stabilized method by Chappelle and Stenberg is: find $w_h \in W_h$ and $\beta_h \in V_h$ such that

$$a^b(\beta_h, \delta) + \frac{E}{h^2 + t^2} \int_\Omega (\nabla w_h - \beta_h) \cdot (\nabla v - \delta) \, dx + \frac{E}{t_*^2} \int_\Omega P_{Q_h}[\nabla w_h - \beta_h] \cdot (\nabla v - \delta) \, dx = \int_\Omega f v \, dx \qquad (1)$$

is true for all $v \in W_h$ and $\delta \in V_h$. Here, the modified thickness t_* is defined by $\frac{1}{t_*^2} = \frac{1}{t^2} - \frac{1}{t^2+h^2}$. Now, we utilize the EAS [AR, BCR, SR] method for computing the projection. Define the co-projection

$$\eta_h = (I - P_{Q_h})[\nabla w_h - \beta_h],$$

which lives in the space

$$\Gamma_h = \{\gamma_h \in [L_2]^2 : \int_\Omega \gamma_h \cdot q_h \, dx = 0 \; \forall q_h \in Q_h, \; \gamma_h|_T \in [P^{p+2}]^2, \gamma_h|_R \in [P^{p+1}]^2\}.$$

The finite element problem (1) can be reformulated as: find $w_h \in W_h$, $\beta_h \in V_h$, and $\eta_h \in \Gamma_h$ such that

$$a^b(\beta_h, \delta) + \frac{E}{t^2} \int_\Omega (\nabla w_h - \beta_h - \eta_h) \cdot (\nabla v - \delta - \gamma) \, dx + \frac{E}{t^2 + h^2} \int_\Omega \eta_h \cdot \gamma \, dx = \int_\Omega f v \, dx \qquad (2)$$

for all $v \in W_h$, $\delta \in V_h$, and $\gamma \in \Gamma_h$. Now, one only has to choose finite element basis functions for the three spaces, and plug them into the standard integration sub-routine.

1.1 Construction of high order finite element spaces

As usual, we define the basis functions of arbitrary order by means of orthogonal polynomials [AS, SDR]. We need Jacobi-polynomials $P_i^{(\alpha,\beta)}(x)$, which are polynomials of order i, and are mutual orthogonal with respect to the weighted inner product $\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta u(x) v(x) \, dx$. Legendre polynomials are included with $P_i(x) = P_i^{(0,0)}(x)$. Orthogonal polynomials can be efficiently evaluated by three-term recurrences. Commonly used are also the integrated Legendre polynomials $L_i(x) := \int_{-1}^{x} P_i(s) \, ds$ for $i \geq 1$, which satisfy $L_i(-1) = L_i(1) = 0$. On triangles one needs the scaled polynomials

$$\widetilde{P}_i^{(\alpha,\beta)}(x, t) = P_i^{(\alpha,\beta)}\Big(\frac{x}{t}\Big) t^i \qquad \widetilde{L}_i(x, t) = L_i\Big(\frac{x}{t}\Big) t^i$$

They are polynomials in x and t, and can be evaluated without division by 3-term recurrences.

The basis functions for continuous 2D elements are either associated with vertices (V), edges (E), or are internal to the elements (I). We specify the basis functions for the vertex $(-1, -1)$, for the edge $(-1, 1) \times \{0\}$, and the internal ones. The remaining ones are obtained by permutation. The indices i and j satisfy $2 \leq i, j \leq p$:

$$\varphi_{V_1} = \frac{1-x}{2} \frac{1-y}{2} \qquad \varphi_{E_1}^j = L_i(x) \frac{1-y}{2} \qquad \varphi_I^{ij} = L_i(x) L_j(y)$$

The shape functions on the triangle [Dub] are defined by means of the barycentric coordinates λ_k. The vertex shape functions are exactly the λ_k. The basis functions on the edge between vertices k and l are

$$\varphi_{E_{kl}}^i = \widetilde{L}_i(\lambda_k - \lambda_j, \lambda_k + \lambda_j), \qquad 2 \leq i \leq p.$$

On the edge E_{kl}, it is L_i, and it vanishes on the other edges. The internal basis functions are

$$\varphi_I^{ij} = \widetilde{L}_i(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2) \lambda_3 P^{(2i+3,2)}(2\lambda_3 - 1), \qquad 2 \leq i, 0 \leq j, i + j \leq p - 1.$$

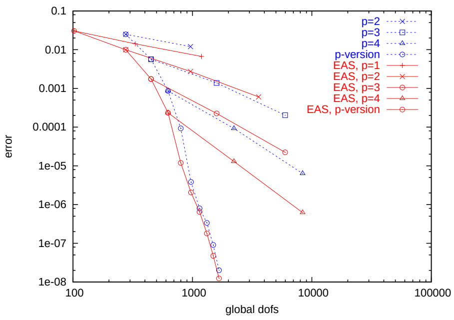

Figure 1: Error in bending stress

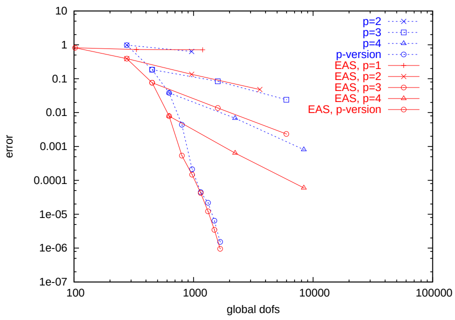

Figure 2: Error in shear stress

These basis functions are used to construct W_h and V_h. The basis functions for the enhanced shear are non-continuous, and must be L_2-orthogonal to lower order polynomials. This is obtained by defining an L_2-orthogonal basis for rectangles and triangles, and taking just the proper components:

$$\begin{aligned} \varphi^{ij}_{\Gamma,R} &= P_i(x)P_j(x) & p-2 \leq \max\{i,j\} \leq p \quad &\text{on rectangles,} \\ \varphi^{ij}_{\Gamma,T} &= \widetilde{P}_i(\lambda_1-\lambda_2, \lambda_1+\lambda_2)P_j^{(0,2i+1)}(2\lambda_3-1) & p-2 \leq i+j \leq p+1 \quad &\text{on triangles.} \end{aligned}$$

1.2 An equal order element

The MITC [BBF] elements have the advantage of requireing the same order for the deflection w and the rotations β. This allows to use them in a 3D formulation for thin structures, where the displacement field is specified in global coordinates rather than in normal and tangential components.

Now, we modify the formulation from above to obtain an equal order EAS element. For this, let $\hat{W}_h = \{w_h \in W_h : w_h|_E \in P^p\}$. The internal order on triangles can be increased to the order of the rotations, if desired. The highest order basis function on each edge was removed. To preserve the approximation property of the shear strain, it is added to the enhanced strains to obtain $\hat{\Gamma}_h = \Gamma_h + \{\nabla \varphi^{p+1}_{E_i}\}$ But now, these edge-based functions are defined element by element, and can be eliminated by static condensation. We show the accuracy of the proposed method for a model problem. Let the plate be the unit square $\Omega = (0,1)^2$, be clamped on the whole boundary, and loaded by a uniform force. Figure 1 and Figure 2 show the L_2-errors of the bending stress and the shear stress, respectively. We compare a conforming method of order

2 A wirebasket preconditioner for high order elements

After static condensation, the global matrix involves coupling of unknowns on edges within the same element, see Figure 3. We propose the following preconditioner: Additionally to assembling Schur complements, assemble inexact Schur complements $\hat{S}$. The $\hat{S}$ is the sum of the 3 (respectively 4) Schur complements related to the edges and adjacent vertices of the element. In this preconditioner, the coupling of the edges on the element is removed, see Figure 4. Thus, the fill in of a sparse factorization of this matrix is much less.

In Figure 5, we study the condition number of this preconditioner in dependency of the polynomial order and the thickness. The condition numbers are uniform in the thickness. In contrast, the same preconditioner behaves similar for thick plates discretized with conforming elements, but degenerates, as the thickness becomes small.

3 Thin 3D structures

We describe a 3D structures by the kinematics of a Naghdi shell. The displacement shape functions are of order p (plus bubbles) in plane, and linear in thickness. Also the enhanced strains are translated from the Reissner Mindlin plate. The strains are transformed to preserve the symmetric tensor. Figure 6

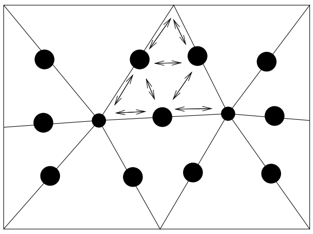

Figure 3: Coupling of Schur system

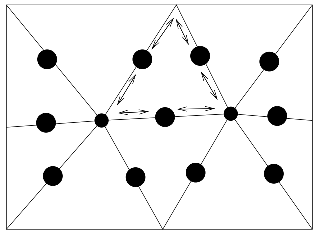

Figure 4: Coupling of wirebasket system

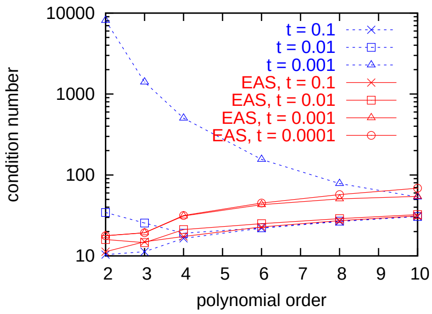

Figure 5: Condition numbers

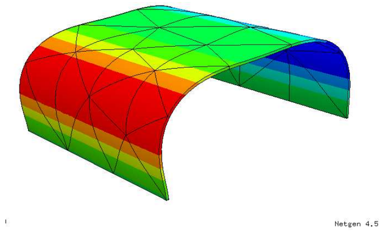

Figure 6: Bending stress

shows the deformation of a bending dominated shell with R=1 and t = 0.01, and polynomial order $p = 6$. Here, the memory needed for the factorization of the wirebasket system took less than 10 percent of the factorization of the Schur complement matrix. The iterative solver needed about 130 iterations, and took less time than the factorization.

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