New applications of hybridization for the Dirichlet problem

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Outline

- Deriving hybridized methods
- A characterization of Lagrange multipliers
- Raviart-Thomas vs. BDM methods
- Preconditioners
- Variable degree methods
- New techniques for error analysis

Mixed methods

Mixed methods for the elliptic problem

Given $f(\boldsymbol{x})$ *and* $g(\boldsymbol{x})$, *find* $u(\boldsymbol{x})$ *such that*

$$-\operatorname{div}\,(\,\boldsymbol{a}(\boldsymbol{x})\,\boldsymbol{\nabla}\,u\,) = f \qquad \text{in } \Omega \subset \mathbb{R}^2$$

$$u = g \qquad \text{on } \partial\Omega$$

are based on its first order reformulation:

Find $u(\boldsymbol{x})$ *and* $\boldsymbol{q}(\boldsymbol{x})$ *such that*

$$\begin{aligned} \boldsymbol{a}(\boldsymbol{x})^{-1}\boldsymbol{q} + \boldsymbol{\nabla}\,u &= 0 \quad && \text{on } \Omega, \\ \operatorname{div}\,\boldsymbol{q} &= f && \text{on } \Omega, \\ u &= g && \text{on } \partial\Omega. \end{aligned}$$

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Mixed methods for the elliptic problem

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$$u = g \qquad \text{on } \partial\Omega$$

are based on its first order reformulation:

Find $u(\boldsymbol{x})$ *and* $\boldsymbol{q}(\boldsymbol{x})$ *such that*

$$\begin{aligned} \boldsymbol{c}(\boldsymbol{x})\,\boldsymbol{q} + \boldsymbol{\nabla}\, u &= 0 \quad \text{on } \Omega, \\ \operatorname{div}\, \boldsymbol{q} &= f \quad \text{on } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

Hybridized mixed methods

$$\boldsymbol{c}\,\boldsymbol{q} + \boldsymbol{\nabla}\, u = 0 \implies$$

$$\int_\tau \boldsymbol{c}\,\boldsymbol{q}\cdot\boldsymbol{v} - \int_\tau u\,\mathrm{div}\,\boldsymbol{v} + \int_{\partial\tau} u\,(\boldsymbol{v}\cdot\boldsymbol{n}) = 0$$

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Hybridized mixed methods

$$\boldsymbol{c}\,\boldsymbol{q} + \boldsymbol{\nabla}\, u = 0 \implies$$

$$\int_\tau \boldsymbol{c}\,\boldsymbol{q}_h \cdot \boldsymbol{v} - \int_\tau u_h \,\mathrm{div}\, \boldsymbol{v} + \int_{\partial\tau\setminus\partial\Omega} \lambda_h (\boldsymbol{v}\cdot\boldsymbol{n}) = \int_{\partial\Omega\cap\partial\tau} g(\boldsymbol{v}\cdot\boldsymbol{n})$$

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$$\boldsymbol{c}\,\boldsymbol{q} + \boldsymbol{\nabla}\, u = 0 \implies$$

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$$\operatorname{div}\, \boldsymbol{q} = f \implies$$

$$\int_\tau w \operatorname{div} \boldsymbol{q} = \int_\tau f\, w$$

Hybridized mixed methods

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Hybridized mixed methods

$$\boldsymbol{c}\,\boldsymbol{q} + \boldsymbol{\nabla}\, u = 0 \;\Longrightarrow$$

$$\int_\tau \boldsymbol{c}\,\boldsymbol{q}_h\cdot\boldsymbol{v} - \int_\tau u_h \operatorname{div}\boldsymbol{v} + \int_{\partial\tau\setminus\partial\Omega} \lambda_h(\boldsymbol{v}\cdot\boldsymbol{n}) = \int_{\partial\Omega\cap\partial\tau} g(\boldsymbol{v}\cdot\boldsymbol{n})$$

$$\operatorname{div}\;\boldsymbol{q} = f \;\Longrightarrow$$

$$\int_\tau w\;\operatorname{div}\boldsymbol{q}_h = \int_\tau f\,w$$

Normal flux continuity $\Longrightarrow$

$$\boldsymbol{q}_h^+\cdot\boldsymbol{n}^+ + \boldsymbol{q}_h^-\cdot\boldsymbol{n}^- \equiv [\![\boldsymbol{q}_h]\!] = 0$$

Hybridized mixed methods

Equations of the hybridized mixed method:

$$\sum_\tau \left(\int_\tau \boldsymbol{c}\, \boldsymbol{q}_h \cdot \boldsymbol{v} - \int_\tau u_h \operatorname{div} \boldsymbol{v} + \int_{\partial\tau \setminus \partial\Omega} \lambda_h \boldsymbol{v} \cdot \boldsymbol{n} \right) = \int_{\partial\Omega} g \boldsymbol{v} \cdot \boldsymbol{n}$$

$$\sum_\tau \left(\int_\tau w \operatorname{div} \boldsymbol{q}_h \right) = \int_\Omega f\, w$$

$$\sum_e \int_e [\![\boldsymbol{q}_h]\!] \mu = 0$$

Discontinuous spaces:

$$\boldsymbol{q}_h|_\tau \in \boldsymbol{R}_k(\tau) \equiv \boldsymbol{x} P_k(\tau) + P_k(\tau)^2, \qquad u_h|_\tau \in P_k(\tau).$$

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$$\sum_{\tau} \left(\int_{\tau} \boldsymbol{c}\, \boldsymbol{q}_h \cdot \boldsymbol{v} - \int_{\tau} u_h \operatorname{div} \boldsymbol{v} + \int_{\partial\tau \backslash \partial\Omega} \lambda_h \boldsymbol{v} \cdot \boldsymbol{n} \right) = \int_{\partial\Omega} g \boldsymbol{v} \cdot \boldsymbol{n}$$

$$\sum_{\tau} \left(\int_{\tau} w \operatorname{div} \boldsymbol{q}_h \right) = \int_{\Omega} f\, w$$

$$\sum_{e} \int_{e} [\![\boldsymbol{q}_h]\!] \mu = 0$$

$\lambda_h =$ *Lagrange multiplier*. λ_h and μ are in

$$M_h = \{\mathrm{m} : \mathrm{m}|_e \in P_k(e) \text{ for all edges } e \text{ of the mesh}\}.$$

Hybridized mixed methods

Equations of the hybridized mixed method:

$$\sum_\tau \left(\int_\tau \boldsymbol{c}\,\boldsymbol{q}_h \cdot \boldsymbol{v} - \int_\tau u_h \operatorname{div} \boldsymbol{v} + \int_{\partial\tau \setminus \partial\Omega} \lambda_h \boldsymbol{v} \cdot \boldsymbol{n} \right) = \int_{\partial\Omega} g \boldsymbol{v} \cdot \boldsymbol{n}$$

$$\sum_\tau \left(\int_\tau w \operatorname{div} \boldsymbol{q}_h \right) = \int_\Omega f\, w$$

$$\sum_e \int_e [\![\boldsymbol{q}_h]\!] \mu = 0$$

It is well known that hybridized mixed method solution components $\boldsymbol{q}_h$ and u_h are the same as the solutions of the mixed method.

Equation for Lagrange multiplier

Operator form of the hybridized mixed method eq.:

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^t & \boldsymbol{C}^t \\ B & D & 0 \\ C & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{q}_h \\ u_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} \boldsymbol{G} \\ F \\ 0 \end{pmatrix}.$$

Eliminate $\boldsymbol{q}_h$ and u_h:

$$\begin{aligned} C\boldsymbol{A}^{-1}(\boldsymbol{A} - B^t(B\boldsymbol{A}^{-1}\boldsymbol{B}^t - D)^{-1}B)\boldsymbol{A}^{-1}\boldsymbol{C}^t\,\lambda_h \\ = -C\boldsymbol{A}^{-1}B^t(B\boldsymbol{A}^{-1}\boldsymbol{B}^t - D)^{-1}(B\boldsymbol{A}^{-1}\boldsymbol{G} - F) \\ + C\boldsymbol{A}^{-1}\boldsymbol{G}, \end{aligned}$$

Just one equation. But pretty complicated.

Why hybridization?

- The equation is symmetric positive definite system, unlike mixed method equations.
- It is a smaller system. (It involves just λ_h, instead of $\boldsymbol{q}_h, u_h,$ and λ_h.)
- Once λ_h is found, $\boldsymbol{q}_h$ and u_h can be found locally (element by element).
- λ_h can be used to get a higher order solution using Arnold-Brezzi postprocessing.
- All advantages of mixed methods. (Solution and flux are approximated simultaneously.)

Perhaps the only disadvantage is that the reduced equation looks complicated.

Remainder

- Deriving hybridized methods
- A characterization of Lagrange multipliers
- Raviart-Thomas vs. BDM methods
- Preconditioners
- Variable degree methods
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A variational characterization

THEOREM. *The Lagrange multiplier* λ_h *is the unique function in* M_h *that satisfies*

$$a_h(\lambda_h, \mu) = b_h(\mu), \quad \text{for all } \mu \in M_h,$$

where the bilinear form and the right hand side are defined by

$$a_h(\eta, \mu) = \int_\Omega \boldsymbol{c}\,(\boldsymbol{\mathcal{Q}}\eta\,)\cdot(\boldsymbol{\mathcal{Q}}\mu\,), \qquad \text{and}$$

$$b_h(\mu) = \int_{\partial\Omega} g\,\boldsymbol{\mathcal{Q}}\mu\,\cdot\boldsymbol{n} + \int_\Omega (\,\mathcal{U}\mu\,)\,f \quad \text{for all } \eta, \mu \in M_h.$$

The lifting map $\mu \mapsto (\,\boldsymbol{\mathcal{Q}}\mu\,,\,\mathcal{U}\mu\,)$ *is defined as follows:*

The lifting maps

Definition of $\mu \mapsto (\boldsymbol{\mathcal{Q}}\mu\,, \mathcal{U}\mu\,)$: On each element τ of the mesh, $\boldsymbol{\mathcal{Q}}\mu\,|_\tau \in \boldsymbol{R}_k(\tau)$ and $\mathcal{U}\mu\,|_\tau \in P_k(\tau)$ are the unique pair of functions that satisfy

$$\int_\tau \boldsymbol{c}\,\boldsymbol{\mathcal{Q}}\mu \cdot \boldsymbol{r} - \int_\tau \mathcal{U}\mu \;\operatorname{div}\, \boldsymbol{r} = -\int_{\partial\tau\setminus\partial\Omega} \mu\,\boldsymbol{r}\cdot\boldsymbol{n},$$

$$\int_\tau w\;\operatorname{div}\,\boldsymbol{\mathcal{Q}}\mu + \int_\tau d\,w\,\mathcal{U}\mu \;= 0,$$

for all $\boldsymbol{r} \in \boldsymbol{R}_k(\tau)$ and $w \in P_k(\tau)$.

Thus, $a_h(\eta, \mu)$ and $b_h(\mu)$ are locally computable.

Lowest order case

Consider the case of the lowest order RT method. Then the lifting of χ_e is

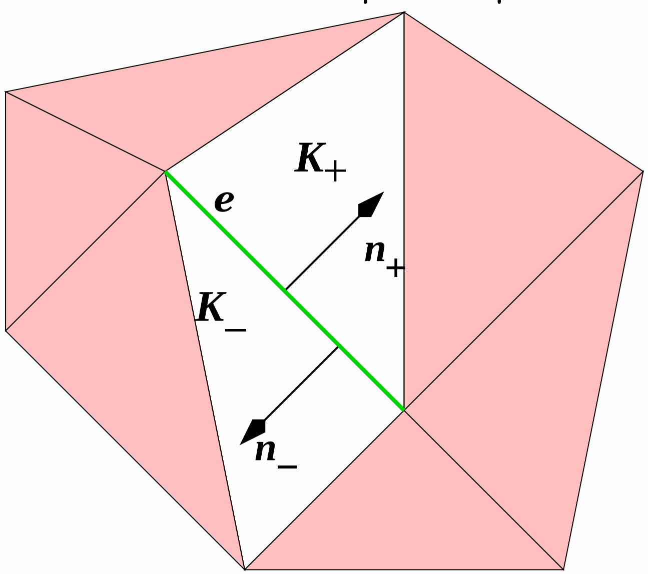
$$\boldsymbol{\mathcal{Q}}\chi_e = \begin{cases} \dfrac{|e|}{|K_+|}\,\tilde{\boldsymbol{c}}_+^{-1}\boldsymbol{n}_+, & \text{in } K_+, \\ \dfrac{|e|}{|K_-|}\,\tilde{\boldsymbol{c}}_-^{-1}\boldsymbol{n}_-, & \text{in } K_-. \end{cases}$$

Notation:

$\chi_e =$ characteristic function of edge e.

$\boldsymbol{n}_\pm =$ unit normals on e.

$$\tilde{\boldsymbol{c}}_\pm = \frac{1}{|K_\pm|}\int_{K_\pm} \boldsymbol{c}\,.$$



$$a_h(\eta,\mu) = \int_\Omega \boldsymbol{c}\,\boldsymbol{\mathcal{Q}}\eta\cdot\boldsymbol{\mathcal{Q}}\mu$$

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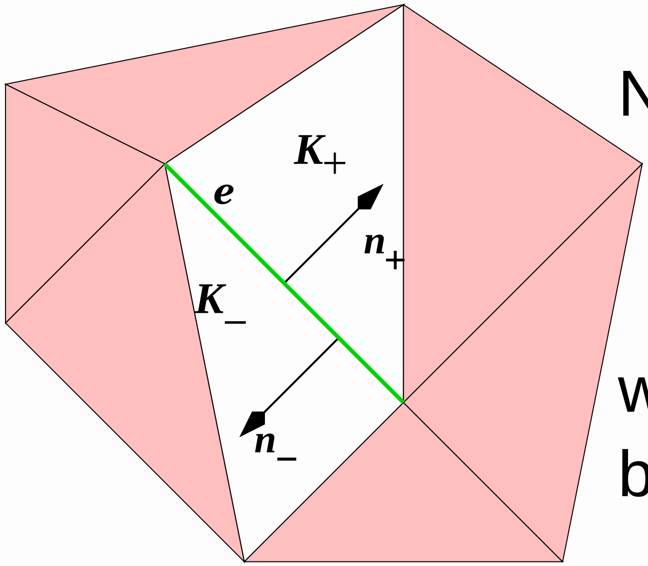

Note that

$$\mathcal{Q}\chi_e = -\tilde{\boldsymbol{c}}_\pm^{-1}\,\boldsymbol{\nabla}\,\psi_e,$$

where $\psi_e =$ P1-nonconforming nodal basis function at e.

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Notation:

$\chi_e =$ characteristic function of edge e.

$\boldsymbol{n}_\pm =$ unit normals on e.

$$\tilde{\boldsymbol{c}}_\pm = \frac{1}{|K_\pm|}\int_{K_\pm} \boldsymbol{c}\,.$$

Now $\boldsymbol{\mathcal{Q}}\chi_e = -\tilde{\boldsymbol{c}}_\pm^{-1}\,\boldsymbol{\nabla}\,\psi_e,$ and $\boldsymbol{c}$ is piecewise constant $\Longrightarrow$

$$a_h(\chi_e,\chi_\ell) = \int_\Omega \boldsymbol{c}\,\boldsymbol{\mathcal{Q}}\chi_e\cdot\boldsymbol{\mathcal{Q}}\chi_\ell = \int_\Omega \boldsymbol{a}\,\boldsymbol{\nabla}\,\psi_e\cdot\boldsymbol{\nabla}\,\psi_\ell.$$

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Notation:

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$\boldsymbol{n}_\pm =$ unit normals on e.

$$\tilde{\boldsymbol{c}}_\pm = \frac{1}{|K_\pm|}\int_{K_\pm} \boldsymbol{c}\,.$$

Thus we rediscover the well known *connection of the lowest order hybridized RT method with the P1-nonconforming method* established by [Arnold & Brezzi, 1985] and [Marini, 1985] by other means.

Local matrix assembly

- Instead of implementing the complicated equation

$$
\begin{aligned}
C\boldsymbol{A}^{-1}(\boldsymbol{A} - B^t(B\boldsymbol{A}^{-1}\boldsymbol{B}^t - D)^{-1}B)\boldsymbol{A}^{-1}\boldsymbol{C}^t\,\lambda_h
&= -C\boldsymbol{A}^{-1}B^t(B\boldsymbol{A}^{-1}\boldsymbol{B}^t - D)^{-1}(B\boldsymbol{A}^{-1}\boldsymbol{G} - F) \\
&\quad + C\boldsymbol{A}^{-1}\boldsymbol{G},
\end{aligned}
$$

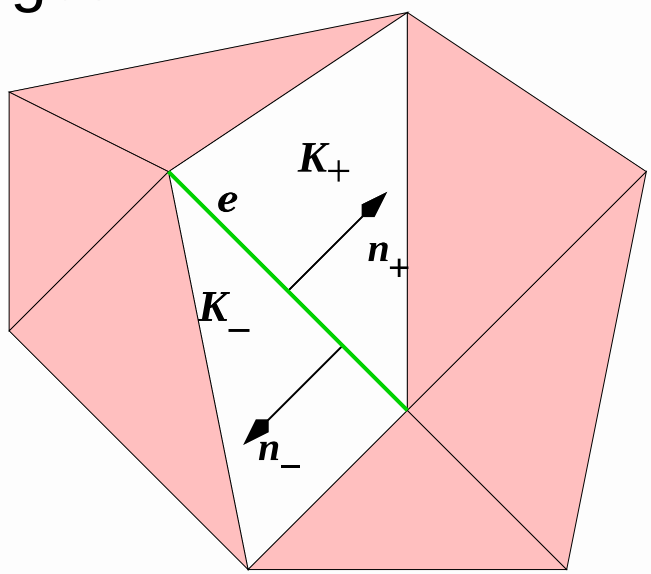
  we can now simply form the “stiffness matrix” of $a_h(\cdot,\cdot)$.

Local matrix assembly

- Liftings can be computed and integrated on the *reference element* and mapped back (using the *Piola map*), so matrix assembly can proceed as in classical finite elements.

Local matrix assembly

- Liftings can be computed and integrated on the *reference element* and mapped back (using the *Piola map*), so matrix assembly can proceed as in classical finite elements.
- A sparse matrix results as every edge can interact with *at most four* other edges.



Remainder

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RT vs. BDM methods

Both methods discretize the same variational equations, but in different discrete spaces.

RT spaces $(k \geq 0)$

Flux: $\boldsymbol{R}_k(\tau)$

Solution: $P_k(\tau)$

Multipliers: $P_k(e)$

Convergence rates:

$$\|\boldsymbol{q}_h - \boldsymbol{q}\|_\Omega = O(h^{k+1})$$

$$\|u_h - u\|_\Omega = O(h^{k+1})$$

$$\|\lambda_h - P_h u\|_{\mathcal{E}_h} = O(h^{k+3/2})$$

(Norms are L^2–norms.)

More accurate u_h.

BDM spaces $(k \geq 1)$

Flux: $(P_k(\tau))^2$

Solution: $P_{k-1}(\tau)$

Multipliers: $P_k(e)$

Convergence rates:

$$\|\boldsymbol{q}_h - \boldsymbol{q}\|_\Omega = O(h^{k+1})$$

$$\|u_h - u\|_\Omega = O(h^k)$$

$$\|\lambda_h - P_h u\|_{\mathcal{E}_h} = O(h^{k+3/2-\delta_{k1}})$$

(Norms are L^2–norms.)

Slightly less expensive.

Identical Lagrange multipliers

Let $\mathit{\Pi}_k$ denote the L^2-orthogonal projection onto the space of functions which are piecewise polynomials of degree k on each triangle of the mesh.

THEOREM. *Suppose f is such that $(\mathit{\Pi}_k - \mathit{\Pi}_{k-1})f = 0$. Consider numerical solution of*

$$\begin{aligned} -\operatorname{div}\big(\boldsymbol{a}(\boldsymbol{x})\,\boldsymbol{\nabla}\,u\big) &= f \quad \text{on } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

by the hybridized RT and BDM methods. Then, the Lagrange multiplier solution components obtained by both methods coincide:

$$\lambda_h^{\mathrm{RT}} = \lambda_h^{\mathrm{BDM}}.$$

Reason for the coincidence

The variational characterization theorem *holds for the BDM method as well*, provided the lifting maps are redefined with respect to the BDM spaces.

Reason for the coincidence

The proof of the theorem proceeds by comparing the left and right hand sides of the discrete variational equations:

$$a_h^{\mathrm{RT}}(\lambda_h^{\mathrm{RT}}, \mu) = b_h^{\mathrm{RT}}(\mu), \quad \text{for all } \mu \in M_h.$$

$$a_h^{\mathrm{BDM}}(\lambda_h^{\mathrm{BDM}}, \mu) = b_h^{\mathrm{BDM}}(\mu), \quad \text{for all } \mu \in M_h.$$

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$$a_h^{\mathrm{BDM}}(\lambda_h^{\mathrm{BDM}},\mu) = b_h^{\mathrm{BDM}}(\mu), \quad \text{for all } \mu \in M_h.$$

The liftings of both methods are divergence free on each element. The divergence free subspaces coincide:

$$\{\boldsymbol{r} \in \boldsymbol{R}_k(\tau) : \operatorname{div} \boldsymbol{r} = 0\} = \{\boldsymbol{p} \in P_k(\tau)^2 : \operatorname{div} \boldsymbol{p} = 0\}$$

$$\Longrightarrow \qquad a_h^{\mathrm{RT}}(\eta,\mu) = a_h^{\mathrm{BDM}}(\eta,\mu).$$

If $(\Pi_k - \Pi_{k-1})f = 0$, then one can show that the right hand sides coincide as well: $b_h^{\mathrm{RT}} = b_h^{\mathrm{BDM}}$.

Further consequences

- Since the stiffness matrices of $a_h^{\mathrm{RT}}(\cdot,\cdot)$ and $a_h^{\mathrm{BDM}}(\cdot,\cdot)$ are identical, any preconditioner for one works for the other as well.

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- There can be no difference in convergence rates for λ_h whenever $(\mathit{\Pi}_k - \mathit{\Pi}_{k-1})f = 0$.

Further consequences

- Since the stiffness matrices of $a_h^{\mathrm{RT}}(\cdot,\cdot)$ and $a_h^{\mathrm{BDM}}(\cdot,\cdot)$ are identical, any preconditioner for one works for the other as well.
- There can be no difference in convergence rates for λ_h whenever $(\mathit{\Pi}_k - \mathit{\Pi}_{k-1})f = 0$.
- It is possible to compute other solution components ($\boldsymbol{q}_h$ and u_h) of both methods *locally* once λ_h is known. Thus it is possible to implement the relatively inexpensive BDM method and recover the higher order RT solution u_h locally.

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Conditioning of the equation

Let

$$\|\lambda\|_h = \left(\sum_\tau h_\tau \|\lambda\|^2_{L^2(\partial\tau)} \right)^{1/2}.$$

THEOREM. *Suppose the mesh is quasiuniform with mesh size h. Then there are positive constants c and C independent of h such that*

$$c\|\lambda\|_h^2 \le a_h(\lambda,\lambda) \le Ch^{-2}\|\lambda\|_h^2, \quad \textit{for all } \lambda \in S_h.$$

Consequently, the spectral condition number of the Lagrange multiplier equation is $O(h^{-2})$.

Earlier preconditioners

- In the lowest order case, [Arnold & Brezzi, 1985] and [Marini, 1985] established that Lagrange multiplier equation is closely related to the P_1-nonconforming method. Preconditioners exploiting this have been devised by [Brenner, 1992], [Chen et al, 1996] and others.

- Preconditioners for systems involving *both* u_h and λ_h have been developed by [Cowsar, Mandel & Wheeler, 1995] and [Rusten & Winther, 1993].

There are no previous works on preconditioning the Lagrange multiplier equation in the higher order case.

Difficulties in preconditioning

- The spectral nature of the higher order Lagrange multiplier equation was perhaps not sufficiently clear before. But it has now become transparent from our variational characterization.

- The mesh dependent nature of $a_h(\cdot, \cdot)$ creates difficulties in multilevel analyses. We overcome this difficulty by techniques similar to those in [Rusten, Vassilevski & Winther, 1996].

- Coarse Lagrange multiplier space is not nested in the fine one. We must construct appropriate intergrid transfer operators.

Additive Schwarz preconditioner

Ω is covered by overlapping subdomains Ω_i satisfying the usual assumptions.

Coarse space (S_H^0): Lowest order Lagrange multipliers on coarse mesh.

Fine space (S_h): Any order Lagrange multiplier space on fine mesh.

A_H: Operator generated by $a_H(\cdot,\cdot)$.

A_i: Operators generated by $a_h(\cdot,\cdot)$ restricted to Ω_i.

Preconditioner: $B_h = \sum_i A_i^{-1} Q_i + I_h A_H^{-1} I_h^t.$

Intergrid transfer operator I_h

Many constructions are possible. A simple strategy:

Let $X_h : S_h^0 \mapsto V_h^0$ be the isomorphism between the P_1-nonconforming space V_h^0 and the lowest order Lagrange multiplier space S_h^0:

$$(X_h \lambda)(x_e) = \lambda|_e,$$

where x_e denotes the mid-point of edge e.

$$\begin{tikzcd}
V_H^0 \arrow[rr, "I_h^V"] \arrow[d, shift right, "X_H^{-1}"'] & & V_h^0 \arrow[d, shift left, "X_h^{-1}"] & \\
S_H^0 \arrow[u, shift right, "X_H"'] \arrow[rrr, bend right, "I_h"'] & & S_h^0 \arrow[u, shift left, "X_h"] \arrow[r, hook] & S_h
\end{tikzcd}$$

$$I_h = X_h^{-1} I_h^V X_H.$$

Uniformity of the preconditioner

THEOREM. *The spectral condition number of the preconditioned operator $B_h A_h$ is bounded independently of coarse and fine mesh sizes h and H.*

We use the following norm equivalence in the proof of the above theorem.

THEOREM. *Let* $m_\tau(\lambda) = \frac{1}{|\partial\tau|}\int_{\partial\tau} \lambda \, ds$ *and*

$$|||\lambda||| = \left(\sum_\tau \frac{1}{h_\tau} \|\lambda - m_\tau(\lambda)\|^2_{L^2(\partial\tau)} \right)^{1/2}.$$

Then,

$$c|||\lambda|||^2 \le a_h(\lambda,\lambda) \le C|||\lambda|||^2, \quad \textit{for all } \lambda \in S_h.$$

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Variable degree methods

Background

For hp adaptivity, one needs to glue together elements with varying polynomial degrees.

Difficulty: One must ensure that varying degrees do not destroy the inf-sup condition. In particular, one needs a projector Π_{hp}^R satisfying a commutativity property:

$$\begin{array}{ccc}
\boldsymbol{H}(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\Pi_{hp}^R \Big\downarrow & & \Pi_{hp} \Big\downarrow \\
\boldsymbol{R}_{hp} & \xrightarrow{\mathrm{div}} & W_{hp}
\end{array}$$

Variable degree methods

Background

For hp adaptivity, one needs to glue together elements with varying polynomial degrees.

First attempt: [Brezzi, Douglas & Marini, 1985].
They gave a projector that satisfies the commuting diagram property whenever the polynomial degrees of adjacent BDM elements vary by at most one.

$$\begin{tikzcd} \boldsymbol{H}(\mathrm{div},\Omega) \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi^R_{hp}"'] & L^2(\Omega) \arrow[d, "\Pi_{hp}"'] \\ \boldsymbol{R}_{hp} \arrow[r, "\mathrm{div}"] & W_{hp} \end{tikzcd}$$

Variable degree methods

Background

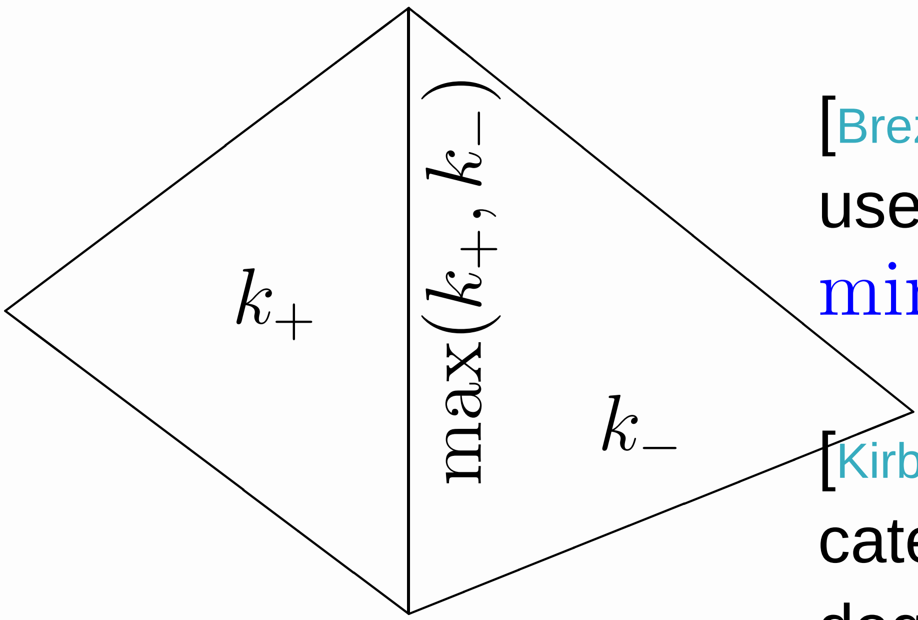
For hp adaptivity, one needs to glue together elements with varying polynomial degrees.

More recently [Demkowicz et al, 2000] gave a projector that satisfies the commutativity properties for BDM spaces of arbitrarily varying polynomial degrees. These ideas apply for RT spaces also.

$$\begin{array}{ccc}
\boldsymbol{H}(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\Pi^R_{hp}\Big\downarrow & & \Pi_{hp}\Big\downarrow \\
\boldsymbol{R}_{hp} & \xrightarrow{\mathrm{div}} & W_{hp}
\end{array}$$

Variable degree via hybridization

We take a different approach via hybridization:
To glue together RT-elements of different degrees k_+ and k_-, use Lagrange multipliers of degree $\max(k_+, k_-)$.



[Brezzi, Douglas & Marini, 1985] used multipliers of degree $\min(k_-, k_+)$.

[Kirby & Dawson, 2003] advocates using multipliers of degree $\max(k_-, k_+)$.

Variable degree RT method

Spaces:

$$\boldsymbol{V}_{hp} = \{\boldsymbol{v} : \quad \boldsymbol{v}|_K \in \boldsymbol{R}_{k(K)} \quad \text{for all elements } K\},$$

$$W_{hp} = \{w : \quad w|_K \in P_{k(K)} \quad \text{for all elements } K\},$$

$$M_{hp} = \{\mu : \quad \mu|_e \in P_{k(e)} \quad \text{for all interior edges } e\},$$

$k(K) =$ degree on element K.
$k(e) =$ max of degrees from either side of e.

Note that functions in these spaces are discontinuous. (The continuity of fluxes are imposed as an equation of the hybridized method.)

Variable degree RT method

Equations:

$$\sum_K \left(\int_K \boldsymbol{c}\, \boldsymbol{q}_{hp} \cdot \boldsymbol{v} - \int_K u_{hp} \operatorname{div} \boldsymbol{v} + \int_{\partial K \setminus \partial \Omega} \lambda_{hp} \boldsymbol{v} \cdot \boldsymbol{n} \right) = \int_{\partial \Omega} g \boldsymbol{v} \cdot \boldsymbol{n}$$

$$\sum_K \left(\int_K w \operatorname{div} \boldsymbol{q}_{hp} \right) = \int_\Omega f\, w$$

$$\sum_e \int_e [\![\boldsymbol{q}_{hp}]\!] \mu = 0$$

This is the same set of equations as before, but now the unknowns and test functions are in the variable degree spaces.

Variable degree RT method

Equations:

$$\sum_K \left(\int_K \boldsymbol{c}\, \boldsymbol{q}_{hp} \cdot \boldsymbol{v} - \int_K u_{hp} \operatorname{div} \boldsymbol{v} + \int_{\partial K \setminus \partial \Omega} \lambda_{hp} \boldsymbol{v} \cdot \boldsymbol{n} \right) = \int_{\partial \Omega} g \boldsymbol{v} \cdot \boldsymbol{n}$$

$$\sum_K \left(\int_K w \operatorname{div} \boldsymbol{q}_{hp} \right) = \int_\Omega f\, w$$

$$\sum_e \int_e [\![\boldsymbol{q}_{hp}]\!] \mu = 0$$

PROPOSITION. *The above system has a unique solution (i.e., the method is well defined).*

Properties

- The Lagrange multipliers has a variational characterization wherein the primal and flux variables do not appear:

$$a_{hp}(\lambda_{hp}, \mu) = b_{hp}(\mu).$$

- Once the Lagrange multiplier system is solved, all the other unknowns can recovered locally:

$$\begin{aligned} \boldsymbol{q}_{hp} &= \mathfrak{Q} g \; + \mathbf{Q} f \; + \boldsymbol{\mathcal{Q}} \lambda_{\,hp}, \\ u_{hp} &= \mathfrak{U} g \; + \mathsf{U} f \; + \mathcal{U} \lambda_{hp}\,. \end{aligned}$$

- We do not need to implement "transition elements" or the "minimum degree rule". It suffices to maintain a set of shape functions on a reference element for every degree.

Remainder

- Deriving hybridized methods
- A characterization of Lagrange multipliers
- Raviart-Thomas vs. BDM methods
- Preconditioners
- Variable degree methods
- New techniques for error analysis

A new error analysis

Idea:

- First, use the variational characterization of λ_h

$$a_h(\lambda_h, \mu) = b_h(\mu)$$

  to obtain an error estimate for λ_h.

- Since the other unknowns are given *locally* by

$$\begin{aligned} \boldsymbol{q}_h &= \mathfrak{Q}g \; + \mathbf{Q}f \; + \boldsymbol{\mathcal{Q}}\lambda_h \,, \\ u_h &= \mathfrak{U}g \; + \mathsf{U}f \; + \mathcal{U}\lambda_h \,. \end{aligned}$$

  the error estimates for $\boldsymbol{q}_h$ and u_h should follow from those of λ_h immediately.

Error estimates

THEOREM.

$$\begin{aligned} \|\lambda_h - P_h u\|_a &\le \|\boldsymbol{q} - \Pi_h^R \boldsymbol{q}\|, \\ \|\boldsymbol{q} - \boldsymbol{q}_h\| &\le \|\boldsymbol{q} - \Pi_h^R \boldsymbol{q}\|. \end{aligned}$$

(Here Π_h^R is the projection in the commuting diagram.)

THEOREM. *Under further regularity assumptions, we have superconvergence:*

$$\|u_h - \Pi_h u\| \le C\,h\Big(\|\Pi_h^R \boldsymbol{q} - \boldsymbol{q}\| + \|\Pi_h u - u\| \Big).$$

(Here Π_h is the $L^2(\Omega)$-orthogonal projection.)

Features of error analysis

- We get previously unknown error estimates for the Lagrange multipliers.
- We start from the Lagrange multiplier and proceed to the other unknowns, in contrast to the usual reverse approach.
- This circumvents the traditional application of inf-sup condition for error analysis (useful for the variable degree case where inf-sup condition was not known).
- One only needs a projector satisfying the commuting diagram property element by element.
- Opens another alternative for hp analysis.

Conclusion

- We developed a variational characterization of Lagrange multipliers that is independent of the other variables, and found it very useful.
- Hybridization is an important practical tool:
  - Effi cient implementation (through smaller, symmetric, and positive defi nite systems)
  - Simplifi es construction of variable degree methods
- Hybridization is a theoretical tool as well:
  - Enhanced accuracy by postprocessing
  - New error analysis techniques
  - Analysis of variable degree methods